

Resonant leading term geometric optics expansions with boundary layers for quasilinear hyperbolic boundary problems

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Abstract

We construct and justify leading order weakly nonlinear geometric optics expansions for nonlinear hyperbolic initial value problems, including the compressible Euler equations. The technique of simultaneous Picard iteration is employed to show approximate solutions tend to the exact solutions in the small wavelength limit. Recent work [2] by Coulombel, Gues, and Williams studied the case of reflecting wave trains whose expansions involve only real phases. We treat generic boundary frequencies by incorporating into our expansions both real and nonreal phases. Nonreal phases introduce difficulties such as approximately solving complex transport equations and result in the addition of boundary layers with exponential decay. This also prevents us from doing an error analysis based on almost-periodic profiles as in [2].

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1 Introduction

In this paper we consider quasilinear hyperbolic fixed boundary problems on the domain $\overline{\mathbb{R}}_+^{d+1} = \{x = (x', x_d) = (t, y, x_d) = (t, x'') : x_d \geq 0\}$ for a class of equations which includes, in particular, the compressible

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Euler equations. The class consists of systems of the following form:

$$\begin{aligned}
(1.1) \quad & \sum_{j=0}^d A_j(v_\epsilon) \partial_{x_j} v_\epsilon = f(v_\epsilon), \\
& b(v_\epsilon)|_{x_d=0} = g_0 + \epsilon G\left(x', \frac{x' \cdot \beta}{\epsilon}\right), \\
& v_\epsilon = u_0 \text{ in } t < 0.
\end{aligned}$$

Here, $A_0 = I$, and we assume $A_j \in C^\infty(\mathbb{R}^N, \mathbb{R}^{N^2})$, $f \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$, and $b \in C^\infty(\mathbb{R}^N, \mathbb{R}^p)$. For the boundary data, we take $G(x', \theta_0) \in C^\infty(\mathbb{R}^d \times \mathbb{T}^1, \mathbb{R}^p)$ periodic in θ_0 and supported in $\{x_0 \geq 0\}$ and the boundary frequency $\beta = (\beta_0, \dots, \beta_{d-1}) \in \mathbb{R}^d \setminus 0$.

We construct leading order weakly nonlinear geometric optics expansions and justify the expansions by establishing their convergence in L^∞ to the unique exact solutions in the large frequency limit, i.e. the small wavelength limit, represented by $\epsilon \rightarrow 0$. This paper completes the treatment of generic β in the following sense: we allow for all β but that in the glancing region, a set of measure zero which is often regarded as a singular case. The case of β belonging to the hyperbolic region, the interior of a cone determined by the A_j , was treated in [2], where the authors constructed leading order expansions of highly oscillatory wavetrains with real phase functions. Our construction includes similar wavetrains with real phases, but in handling more general β we are required to add to the expansions exponentially decaying oscillations, which we refer to as *elliptic boundary layers*, featuring nonreal phase functions.

1.1 The problem and its reformulations

First we motivate and outline the relevant reformulations of the problem (1.1) and the form of the expansion.

1.1.1 The solution as a perturbation

Seeking $v_\epsilon = u_0 + \epsilon u_\epsilon$, a perturbation of a constant state u_0 with $f(u_0) = 0$ and $b(u_0) = g_0$, we translate (1.1) to the following problem for u_ϵ (where the A_j have been altered accordingly):

$$\begin{aligned}
(1.2) \quad & (a) \quad P(\epsilon u_\epsilon, \partial_x) u_\epsilon := \sum_{j=0}^d A_j(\epsilon u_\epsilon) \partial_{x_j} u_\epsilon = \mathcal{F}(\epsilon u_\epsilon) u_\epsilon, \\
& (b) \quad B(\epsilon u_\epsilon) u_\epsilon|_{x_d=0} = G\left(x', \frac{x' \cdot \beta}{\epsilon}\right), \\
& (c) \quad u_\epsilon = 0 \text{ in } t < 0.
\end{aligned}$$

where $B(v)$ and $\mathcal{F}(v)$ denote the $p \times N$ and $N \times N$ real matrices, C^∞ in v , defined by

$$(1.3) \quad B(v)v = b(u_0 + v) - b(u_0), \quad \mathcal{F}(v)v = f(u_0 + v).$$

Let us define the operator

$$(1.4) \quad L(\partial_x) := P(0, \partial_x) = \partial_t + \sum_{j=1}^d A_j(0) \partial_{x_j}.$$

We assume that $L(\partial_x)$ is hyperbolic with characteristics of constant multiplicity, making the following:

Assumption 1.1. *The matrix $A_0 = I$. For an open neighborhood \mathcal{O} of $0 \in \mathbb{R}^N$, there exists an integer $q \geq 1$, some real functions $\lambda_1, \dots, \lambda_q$ that are C^∞ on $\mathcal{O} \times \mathbb{R}^d \setminus \{0\}$, homogeneous of degree 1, and analytic in ξ , and there exist some positive integers ν_1, \dots, ν_q such that:*

$$(1.5) \quad \det \left[\tau I + \sum_{j=1}^d \xi_j A_j(u) \right] = \prod_{k=1}^q (\tau + \lambda_k(u, \xi))^{\nu_k}$$

for $u \in \mathcal{O}$, $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \setminus \{0\}$. Moreover the eigenvalues $\lambda_1(u, \xi), \dots, \lambda_q(u, \xi)$ are semi-simple (their algebraic multiplicity equals their geometric multiplicity) and satisfy $\lambda_1(u, \xi) < \dots < \lambda_q(u, \xi)$ for all $u \in \mathcal{O}$, $\xi \in \mathbb{R}^d \setminus \{0\}$.

Also, for our study we consider a noncharacteristic boundary $\{x_d = 0\}$:

Assumption 1.2. For $u \in \mathcal{O}$ the matrix $A_d(u)$ is invertible and the matrix $B(u)$ has maximal rank, its rank p being equal to the number of positive eigenvalues of $A_d(u)$ (counted with their multiplicity).

We now provide some notation regarding frequencies $\tau - i\gamma \in \mathbb{C}$ and $\eta \in \mathbb{R}^{d-1}$ dual to the variables t and y , respectively. We define the matrix

$$(1.6) \quad \mathcal{A}(\zeta) := -i A_d^{-1}(0) \left(\tau I + \sum_{j=1}^{d-1} \eta_j A_j(0) \right), \quad \zeta := (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1},$$

and the sets of frequencies

$$\begin{aligned} \Xi &:= \left\{ (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus (0, 0) : \gamma \geq 0 \right\}, & \Sigma &:= \left\{ \zeta \in \Xi : \tau^2 + \gamma^2 + |\eta|^2 = 1 \right\}, \\ \Xi_0 &:= \left\{ (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} \setminus (0, 0) \right\} = \Xi \cap \{\gamma = 0\}, & \Sigma_0 &:= \Sigma \cap \Xi_0. \end{aligned}$$

With these in mind, we define the symbol

$$(1.7) \quad L(\tau, \xi) := \tau I + \sum_{j=1}^d \xi_j A_j(0).$$

The following is a result due to Kreiss [5] in the case of strict hyperbolicity, i.e. when all the eigenvalues in Assumption 1.1 have multiplicity $\nu_j = 1$, and to Métivier [8] in our more general case.

Proposition 1.3 ([5, 8]). *Let Assumptions 1.1 and 1.2 be satisfied. Then for all $\zeta \in \Xi \setminus \Xi_0$, the matrix $\mathcal{A}(\zeta)$ has no purely imaginary eigenvalue and its stable subspace $\mathbb{E}^s(\zeta)$ has dimension p . Furthermore, \mathbb{E}^s defines an analytic vector bundle over $\Xi \setminus \Xi_0$ that can be extended as a continuous vector bundle over Ξ .*

For $(\tau, \eta) \in \Xi_0$, we define $\mathbb{E}^s(\tau, \eta)$ to be the continuous extension obtained in Proposition 1.3 of \mathbb{E}^s to (τ, η) .

Now we describe our assumption of *uniform stability*, defined as in [5, 3]:

Definition 1.4. *The problem (1.2) is uniformly stable at $u = 0$ if the linearized operators $(L(\partial_x), B(0))$ at $u = 0$ are such that*

$$(1.8) \quad B(0) : \mathbb{E}^s(\tau - i\gamma, \eta) \rightarrow \mathbb{C}^p \text{ is an isomorphism for all } (\tau - i\gamma, \eta) \in \Sigma.$$

Assumption 1.5. *The problem (1.2) is uniformly stable at $u = 0$.*

We now discuss an important example, the Euler equations, which satisfies the above assumptions so that our main results may be applied. We remark, however, that we also require Assumption 1.7, an assumption on the boundary frequency β , which is explained in Section 1.1.2.

Example 1.6 (Euler equations). *The following are the isentropic, compressible Euler equations in three space dimensions on the half space $\{x_3 \geq 0\}$, in the unknowns density ρ and velocity $u = (u_1, u_2, u_3)$:*

$$(1.9) \quad \partial_t \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \end{pmatrix} + \partial_{x_1} \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + p(\rho) \\ \rho u_2 u_1 \\ \rho u_3 u_1 \end{pmatrix} + \partial_{x_2} \begin{pmatrix} \rho u_2 \\ \rho u_1 u_2 \\ \rho u_2^2 + p(\rho) \\ \rho u_3 u_2 \end{pmatrix} + \partial_{x_3} \begin{pmatrix} \rho u_3 \\ \rho u_1 u_3 \\ \rho u_2 u_3 \\ \rho u_3^2 + p(\rho) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $p(\rho)$ is the pressure. The hyperbolicity assumption, Assumption 1.1, is satisfied in the region of state space where $\rho > 0$, $c^2 = p'(\rho) > 0$. In this case the eigenvalues $\lambda_k(\rho, u, \xi)$ are

$$(1.10) \quad \lambda_1 = u \cdot \xi - c|\xi|, \quad \lambda_2 = u \cdot \xi, \quad \lambda_3 = u \cdot \xi + c|\xi|, \quad \text{with } (\nu_1, \nu_2, \nu_3) = (1, 2, 1).$$

For this problem, the boundary condition we impose is the natural “residual boundary condition,” which is obtained in the vanishing viscosity limit of the compressible Navier-Stokes equations with Dirichlet boundary conditions. Fixing any constant state (ρ, u) with $u_3 \notin \{0, -c, c\}$ about which to linearize the problem, so that $(\rho, u) = u_0$ in (1.1), we have noncharacteristic boundary $\{x_3 = 0\}$ for the system. Consider in particular Assumption 1.5 in each of the following cases:

(a) **Subsonic outflow:** $u_3 < 0$, $|u_3| < c$. In this case there is exactly one positive eigenvalue of $A_3(\rho, u)$ ($p = 1$), so we need one scalar boundary condition. Taking $b(\rho, u) = u_3$ in (1.1), we have $B(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$ and Assumption 1.5 is satisfied.

(b) **Subsonic inflow:** $0 < u_3 < c$. For this we get $p = 3$ and boundary condition $b(\rho, u) = (\rho u_3, u_1, u_2)$ and linearized operator

$$(1.11) \quad B(0)(\dot{\rho}, \dot{u}) = (\dot{\rho}u_3 + \rho\dot{u}_3, \dot{u}_1, \dot{u}_2),$$

which satisfies Assumption 1.5.

(c) **Supersonic inflow:** $0 < c < u_3$. This case is trivial, with $p = 4$ and $B(0)$ the 4×4 identity matrix, and so Assumption 1.5 holds.

(d) **Supersonic outflow:** $u_3 < 0$, $|u_3| > c$. This is another trivial case, with $p = 0$, where $B(0)$ is absent, meaning Assumption 1.5 holds vacuously.

With clear modifications of the above statements, the same holds for the 2D Euler equations, which are in fact strictly hyperbolic. For complete proofs and discussion verifying these cases, we refer the reader to [4], Section 5.

To study geometric optics for nonlinear problems, it is important to establish the existence of exact solutions on a fixed time interval independent of the wavelength ϵ . For the system (1.2) and thus (1.1), this was proved by M. Williams in [11] through the use of the singular system, which we discuss further in section 1.1.3.

1.1.2 The approximate solution constructed in terms of an M -periodic function

The goal of this paper is to obtain qualitative information about the exact solution to (1.2) by explicitly constructing an approximate solution with a geometric optics expansion that exhibits the qualitative information, and then showing that this approximate solution tends to the exact solution as $\epsilon \rightarrow 0$.

Regular boundary frequencies.

We now discuss an assumption regarding the boundary frequency $\beta = (\underline{\tau}, \underline{\eta}) \in \mathbb{R} \times \mathbb{R}^{d-1} \setminus (0, 0)$ which is relevant to the construction of the approximate solution. The planar real phase ϕ_0 is defined on the boundary by

$$(1.12) \quad \phi_0(x') := \underline{\tau}t + \underline{\eta} \cdot y = \beta \cdot x'.$$

Note the relationship to the boundary data (1.2)(b). The oscillations on the boundary associated to ϕ_0 result in oscillations associated to some planar phases ϕ_m , which are characteristic for $L(\partial_x)$ and which have trace on the boundary equal to ϕ_0 . Such oscillations largely motivate the ansatz for our approximate solution, dependent on the planar phases ϕ_m . The following assumption allows us to obtain the ϕ_m .

Assumption 1.7. For $\zeta = (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1}$ consider the matrix

$$(1.13) \quad \frac{1}{i} \mathcal{A}(\tau - i\gamma, \eta) = -A_d^{-1}(0) \left((\tau - i\gamma) I + \sum_{j=1}^{d-1} \eta_j A_j(0) \right).$$

Let $\underline{\omega}_m$, $m = 1, \dots, M$ denote the distinct eigenvalues of $\frac{1}{i} \mathcal{A}(\underline{\tau}, \underline{\eta})$. We suppose that each $\underline{\omega}_m$ is a semisimple eigenvalue with multiplicity denoted by μ_m . Moreover, we assume there is a conic neighborhood \mathcal{O} of β in $\mathbb{C} \times \mathbb{R}^{d-1} \setminus \{0\}$ on which the eigenvalues of $\frac{1}{i} \mathcal{A}(\zeta)$ are semisimple and given by smooth functions $\omega_m(\zeta)$, $m = 1, \dots, M$ where $\underline{\omega}_m = \omega_m(\beta)$ and $\omega_m(\zeta)$ is of constant multiplicity μ_m .

Remark 1.8. Using Assumption 1.7, one can show that the smooth functions $\omega_m(\zeta)$ are in fact analytic on a conic neighborhood of β . In particular, the analyticity is used to prove Lemma 1.13.

We call β a *regular boundary frequency* provided Assumption 1.7 holds. Now, we define the phase functions

$$(1.14) \quad \phi_m(x) := \phi_0(x') + \underline{\omega}_m x_d = (\beta, \underline{\omega}_m) \cdot x,$$

and we have the complex characteristic vector field associated to ϕ_m :

$$(1.15) \quad X_{\phi_m} := \partial_{x_d} + \sum_{j=0}^{d-1} -\partial_{\xi_j} \omega_m(\beta) \partial_{x_j},$$

which is real for real $\underline{\omega}_m$. Moreover, for each $\underline{\omega}_m$ which is real, Assumption 1.1 implies there is a unique $k_m \in \{1, \dots, q\}$ such that $\tau + \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) = 0$, and in this case we have $\mu_m = \nu_{k_m}$.¹ With this in mind, we make the following definition:

Definition 1.9. (i) For m such that $\underline{\omega}_m$ is real, we call $(\beta, \underline{\omega}_m)$ a *hyperbolic mode* if

$$(1.16) \quad \partial_{\xi_d} \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) \neq 0.$$

(ii) For m with nonreal $\underline{\omega}_m$, we call $(\beta, \underline{\omega}_m)$ an *elliptic mode*.

Remark 1.10. (i) For each real $\underline{\omega}_m$, Assumption 1.7 guarantees that $(\beta, \underline{\omega}_m)$ is a hyperbolic mode. That (1.16) holds is a consequence of the semisimplicity of $\underline{\omega}_m$.² Also, the condition (1.16) and the implicit function theorem imply that ω_m is real and of multiplicity ν_{k_m} in a neighborhood of β . Thus, there is a slight redundancy in Assumption 1.7 when $\underline{\omega}_m$ is real.

(ii) When $\underline{\omega}_m$ is nonreal, the fact that the A_j are real implies that $\frac{1}{\tau} \mathcal{A}(\beta)$ also has the complex conjugate $\overline{\underline{\omega}_m}$ as an eigenvalue, and thus $\overline{\underline{\omega}_m} = \underline{\omega}_{m'}$ for some $m' \neq m$. Furthermore, if the vector $r \in \mathbb{C}^N$ is an eigenvector of $\frac{1}{\tau} \mathcal{A}(\beta)$ associated to $\underline{\omega}_m$, then \bar{r} is an eigenvector associated to $\overline{\underline{\omega}_m} = \underline{\omega}_{m'}$.

(iii) The boundary frequency β lies in the hyperbolic region (as defined in [2]) if and only if Assumption 1.7 holds with all $\underline{\omega}_m$ real. The elliptic region consists of all β such that Assumption 1.7 holds with all $\underline{\omega}_m$ nonreal.

Example 1.11. We return to the 3D Euler equations, linearizing about $(\rho, u) = (\rho, u_1, u_2, u_3)$, as in Example 1.6.

For $|u_3| > c$ the set of regular boundary frequencies is all of $\mathbb{R}^3 \setminus 0$. In the case $|u_3| < c$, one finds that the set of regular boundary frequencies is

$$(1.17) \quad \left\{ (\tau, \eta) \in \mathbb{R}^3 : |\tau + u_1 \eta_1 + u_2 \eta_2| \neq \sqrt{c^2 - u_3^2} |\eta| \right\}.$$

We remark that in the former case, the hyperbolic region is also $\mathcal{H} = \mathbb{R}^3 \setminus 0$, and in the latter case $\mathcal{H} = \left\{ (\tau, \eta) \in \mathbb{R}^3 : |\tau + u_1 \eta_1 + u_2 \eta_2| > \sqrt{c^2 - u_3^2} |\eta| \right\}$.

For real $\underline{\omega}_m$ we have the associated real group velocity:

$$(1.18) \quad \mathbf{v}_m := \nabla \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m),$$

which has the following relationship with the characteristic vector field (1.15):

$$(1.19) \quad \partial_{\xi_0} \omega_m(\beta) = -\frac{1}{\partial_{\xi_d} \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m)}, \quad \partial_{\xi_j} \omega_m(\beta) = -\frac{\partial_{\xi_j} \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m)}{\partial_{\xi_d} \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m)}, \quad j = 1, \dots, d-1.$$

Observe that each group velocity \mathbf{v}_m can be thought of as either incoming or outgoing with respect to the interior of the domain \mathbb{R}_+^d , in particular since the last coordinate of \mathbf{v}_m is nonzero, by (1.16). With this in mind, we classify the phases in the following way:

¹Assumption 1.1 has no immediate implication for the multiplicity of nonreal $\underline{\omega}_m$.

²If $\underline{\omega}_m$ is real and $\partial_{\xi_d} \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) = 0$, we refer to $(\beta, \underline{\omega}_m)$ as a *glancing mode*. An explanation of how (1.16) follows from semisimplicity can be found in the proof of Lemma 2.7 of [8].

Definition 1.12. For real $\underline{\omega}_m$, the phase ϕ_m is incoming if the group velocity \mathbf{v}_m is incoming (that is, $\partial_{\xi_d} \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) > 0$), and it is outgoing if the group velocity \mathbf{v}_m is outgoing ($\partial_{\xi_d} \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) < 0$).

With this classification of the real phases, we let \mathcal{I} denote the set of indices $m \in \{1, \dots, M\}$ such that ϕ_m is incoming and \mathcal{O} the set of m such that ϕ_m is outgoing. We classify the remaining complex phases ϕ_m , which correspond to nonreal $\underline{\omega}_m$, by the distinction that \mathcal{P} is the set of m such that $\text{Im } \underline{\omega}_m > 0$ and \mathcal{N} is the set of m such that $\text{Im } \underline{\omega}_m < 0$. We thus form the partition

$$(1.20) \quad \{1, \dots, M\} = \mathcal{I} \cup \mathcal{O} \cup \mathcal{P} \cup \mathcal{N}.$$

Recall the $\omega_m(\tau, \eta)$ are eigenvalues of $\frac{1}{i} \mathcal{A}(\tau, \eta)$, meaning the $i\omega_m(\tau, \eta)$ are the eigenvalues of $\mathcal{A}(\tau, \eta)$. Consider also the fact that $\text{Im } \omega_m(\tau, \eta) > 0$ if and only if $\text{Re } i\omega_m(\tau, \eta) < 0$. From these observations, we see that the stable subspace $\mathbb{E}^s(\tau, \eta)$ of $\mathcal{A}(\tau, \eta)$ must contain each of the eigenspaces corresponding to some $\omega_m(\tau, \eta)$ with $\text{Re } i\omega_m < 0$, so that $\mathbb{E}^s(\underline{\tau}, \underline{\eta})$ contains the subspaces $\text{Ker } L(d\phi_m)$ for $m \in \mathcal{P}$. The incoming phases, corresponding to $m \in \mathcal{I}$, also play a role in setting up a decomposition for the stable subspace \mathbb{E}^s at the boundary, which is established in the following lemma.

Lemma 1.13. The stable subspace $\mathbb{E}^s(\underline{\tau}, \underline{\eta})$ admits the decomposition:

$$(1.21) \quad \mathbb{E}^s(\underline{\tau}, \underline{\eta}) = \oplus_{m \in \mathcal{I} \cup \mathcal{P}} \text{Ker } L(d\phi_m),$$

where, in the decomposition, the vector spaces $\text{Ker } L(d\phi_m)$ for $m \in \mathcal{I}$ admit a basis of real vectors.

Proof. It is easy to show that the subspaces $\text{Ker } L(d\phi_m)$ for $m \in \mathcal{P}$ are in $\mathbb{E}^s(\underline{\tau}, \underline{\eta})$. We will show that this is also the case for the subspaces $\text{Ker } L(d\phi_m)$ with $m \in \mathcal{I}$, from which the result will follow. Since $\mathbb{E}^s(\underline{\tau}, \underline{\eta})$ is close to $\mathbb{E}^s(\underline{\tau} - i\gamma, \underline{\eta})$ for small $\gamma > 0$, in accordance with Proposition 1.3, it will suffice to show for $m \in \mathcal{I}$ that $i\omega_m(\underline{\tau} - i\gamma, \underline{\eta})$, close to $i\underline{\omega}_m$, satisfies

$$(1.22) \quad \text{Re } i\omega_m(\underline{\tau} - i\gamma, \underline{\eta}) < 0.$$

Recalling (1.19), we see that since ϕ_m is incoming, i.e. $m \in \mathcal{I}$, we have

$$(1.23) \quad \partial_{\zeta_0} \omega_m(\underline{\tau}, \underline{\eta}) < 0.$$

Using analyticity of $\omega_m(\zeta)$ with (1.23), since $\omega_m(\underline{\tau}, \underline{\eta})$ has imaginary part equal to zero, it follows that $\omega_m(\underline{\tau} - i\gamma, \underline{\eta})$ has positive imaginary part for small positive γ . Therefore (1.22) holds. The statement that each of the vector spaces $\text{Ker } L(d\phi_m)$ for $m \in \mathcal{I}$ admits a basis of real vectors follows from Lemma 1.14, which was proved in [1]. \square

The ansatz.

The following lemma, proved in [1], provides a decomposition which is key to the construction of our ansatz and will serve us later in the construction of the projectors which give us the profile equations.

Lemma 1.14. The space \mathbb{C}^N admits the decomposition:

$$(1.24) \quad \mathbb{C}^N = \oplus_{m=1}^M \text{Ker } L(d\phi_m)$$

and each vector space in (1.24) with $m \in \mathcal{I} \cup \mathcal{O}$ admits a basis of real vectors. If we let P_1, \dots, P_M denote the projectors associated with the decomposition (1.24), then for all $m = 1, \dots, M$, there holds $\text{Im } A_d^{-1} L(d\phi_m) = \text{Ker } P_m$.

For each $m = 1, \dots, M$, we choose a basis of $\text{Ker } L(d\phi_m)$:

$$(1.25) \quad r_{m,k}, \quad k = 1, \dots, \mu_m,$$

where for real $\underline{\omega}_m$, we take a basis of real $r_{m,k}$. Now we are in a position to give the ansatz for our approximate solution to (1.2).

$$(1.26) \quad u_\epsilon^a(x) = \underline{v}(x) + \sum_{m=1}^M \sum_{k=1}^{\mu_m} \sigma_{m,k} \left(x, \frac{\phi_m(x)}{\epsilon} \right) r_{m,k},$$

where $\underline{v}(x)$ and the $\sigma_{m,k}(x, \theta_m)$ are C^1 functions. Additionally, we require that each of the $\sigma_{m,k}(x, \theta_m)$ is periodic in θ_m with mean 0, and we will refer to these as the *periodic profiles*. With these, we plug (1.26) into (1.2)(a) and expand the error in ascending powers of ϵ :

$$(1.27) \quad P(\epsilon u_\epsilon^a, \partial_x) u_\epsilon^a - \mathcal{F}(\epsilon u_\epsilon^a) u_\epsilon^a = \epsilon^{-1} \left(\sum_{m=1}^M \sum_{k=0}^{\mu_m} L(d\phi_m) \partial_{\theta_m} \sigma_{m,k} \left(x, \frac{\phi_m(x)}{\epsilon} \right) r_{m,k} \right) + \epsilon^0 (\dots) + \dots$$

Observe that since each $r_{m,k}$ belongs to $\text{Ker } L(d\phi_m)$, the term of order $\frac{1}{\epsilon}$ in (1.27) is in fact zero.

Equation (1.26) is a special case of a substitution of the form

$$(1.28) \quad u_\epsilon^a(x) = \mathcal{V}^0 \left(x, \frac{\phi_1(x)}{\epsilon}, \dots, \frac{\phi_M(x)}{\epsilon} \right),$$

for a function $\mathcal{V}^0(x, \theta_1, \dots, \theta_M)$ periodic in $(\theta_1, \dots, \theta_M)$. One finds that solving (1.2)(a) to order $\frac{1}{\epsilon}$ amounts to satisfying the condition

$$(1.29) \quad \mathcal{L}(\partial_\theta) \mathcal{V}^0 := \sum_{m=1}^M \tilde{L}(d\phi_m) \partial_{\theta_m} \mathcal{V}^0 = 0,$$

where $\tilde{L}(d\phi_m) = A_d^{-1} L(d\phi_m)$. Indeed, (1.29) is satisfied if \mathcal{V}^0 has the form

$$(1.30) \quad \mathcal{V}^0(x, \theta_1, \dots, \theta_M) = \underline{v}(x) + \sum_{m=1}^M \sum_{k=1}^{\mu_m} \sigma_{m,k}(x, \theta_m) r_{m,k},$$

which is consistent with the ansatz of (1.26).

One can try to solve (1.2)(a) to higher order by constructing a *corrected approximate solution* u_ϵ^c which, upon replacing u_ϵ^a in (1.27), results in the vanishing of the coefficients of higher powers of ϵ . In fact, we will obtain more conditions on \underline{v} and the $\sigma_{m,k}$ which will ultimately determine our ansatz for u_ϵ^a , by considering the existence of such a u_ϵ^c which corrects u_ϵ^a and agrees in the leading term. Agreement is meant in the sense that, for some M -periodic $\mathcal{V}^1(x, \theta_1, \dots, \theta_M)$, we have

$$(1.31) \quad u_\epsilon^c(x) = u_\epsilon^a(x) + \epsilon \mathcal{V}^1 \left(x, \frac{\phi_1(x)}{\epsilon}, \dots, \frac{\phi_M(x)}{\epsilon} \right) = \mathcal{V}^0 \left(x, \frac{\phi_1(x)}{\epsilon}, \dots, \frac{\phi_M(x)}{\epsilon} \right) + \epsilon \mathcal{V}^1 \left(x, \frac{\phi_1(x)}{\epsilon}, \dots, \frac{\phi_M(x)}{\epsilon} \right).$$

Here \mathcal{V}^1 is not generally expected to share the form of \mathcal{V}^0 seen in (1.30), as we do not expect \mathcal{V}^1 to solve (1.29). The function \mathcal{V}^1 is determined by other differential equations arising from setting higher order terms of (1.27) to zero.

Remark 1.15. Ensuring the possibility of constructing such u_ϵ^c , which suggests we have the right leading term u_ϵ^a , requires *solvability conditions* to hold; these will be translated into conditions known as the *profile equations* on the *profiles*, that is, $\underline{v}(x)$ and the $\sigma_{m,k}(x, \theta_m)$. We will discuss the profile equations in greater depth later.

Expansions with elliptic boundary layers were constructed in [10], treating the semilinear problem for generic β ,³ and in [6], where a semilinear dispersive problem with maximally dissipative boundary conditions was considered. In both papers, higher order expansions in ascending powers of ϵ , such as

$$(1.32) \quad \mathcal{V}^0 \left(x, \frac{\phi_1(x)}{\epsilon}, \dots, \frac{\phi_M(x)}{\epsilon} \right) + \dots + \epsilon^N \mathcal{V}^N \left(x, \frac{\phi_1(x)}{\epsilon}, \dots, \frac{\phi_M(x)}{\epsilon} \right),$$

are used to justify the approximate solution as well as construct the exact solution. High order expansions involving surface waves, which arise from a failure of the uniform Lopatinski condition at a frequency β in the elliptic region, were constructed and justified in [7] for quasilinear hyperbolic systems. Here in the spirit of [2] we justify a leading term expansion for solutions to (1.1) oscillating with multiple phases. In this situation it is impossible to construct a high order expansion without a small divisor assumption, an assumption we do not make. While the small divisor assumption would exclude β only from a set of measure zero, verifying it for a given β can be difficult if not impossible.

³In fact, [10] also treats β resulting in glancing modes of order two, which do not result in blow-up, unlike higher-order glancing cases.

1.1.3 The solution of the singular system and the main theorem

The following theorem verifies that the approximate solution is in fact close to the exact solution for small ϵ .

Theorem 1.16. *Suppose the assumptions hold that $L(\partial_x)$ is hyperbolic with characteristics of constant multiplicity, the boundary $\{x_d = 0\}$ is noncharacteristic, $(L(\partial_x), B(0))$ satisfies the uniform stability condition, and β is a regular boundary frequency (i.e. Assumptions 1.1, 1.2, 1.5, 1.7, resp.) Then given the exact solution u_ϵ of (1.2) defined on a time interval $(-\infty, T]$ independent of ϵ , for u_ϵ^a as in (1.26), where \underline{v} and the $\sigma_{m,k}$ satisfy⁴ the profile equations, defined on a time interval $(-\infty, T']$ independent of ϵ , we have*

$$(1.33) \quad |u_\epsilon(x) - u_\epsilon^a(x)|_{L^\infty((-\infty, T_0] \times \overline{\mathbb{R}}_+^d)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0,$$

where T_0 is the minimum of T and T' .

Observe for this result that we need in particular a positive lower bound on the existence time T_ϵ of the exact solution u_ϵ to (1.2) which holds uniformly for small ϵ . That is, for some positive ϵ_0 ,

$$(1.34) \quad \inf_{(0, \epsilon_0]} T_\epsilon > 0.$$

However, applying the standard theory to the problem (1.2) yields existence times T_ϵ which shrink to zero as the Sobolev norms of the initial data blow up in the limit $\epsilon \rightarrow 0$. To get estimates uniform in ϵ , one may use a reformulation of the system (1.2) known as the *singular system*, which is also used in the framework of [2]. In fact, to prove the main theorem, we show a stronger result involving the solution of the singular system. The singular system is obtained by recasting (1.2) in terms of an unknown $U_\epsilon(x, \theta_0)$ periodic in θ_0 such that a solution of (1.2) is to be formed by making the substitution

$$(1.35) \quad u_\epsilon(x) = U_\epsilon \left(x, \frac{\beta \cdot x'}{\epsilon} \right).$$

The singular system is written in the form

$$(1.36) \quad \begin{aligned} (a) \quad & \partial_{x_d} U_\epsilon + \sum_{j=0}^{d-1} \tilde{A}_j(\epsilon U_\epsilon) \left(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\epsilon} \right) U_\epsilon = F(\epsilon U_\epsilon) U_\epsilon, \\ (b) \quad & B(\epsilon U_\epsilon)(U_\epsilon)|_{x_d=0} = G(x', \theta_0), \\ (c) \quad & U_\epsilon = 0 \text{ in } t < 0, \end{aligned}$$

where $F = A_d^{-1} \mathcal{F}$. Indeed, the main benefit of expressing the problem in terms of the new unknown U_ϵ is that it is possible to obtain estimates uniform in ϵ for U_ϵ and thus existence of the solution u_ϵ to (1.2) in a fixed time interval independent of ϵ , demonstrated in [11] by M. Williams, where he constructed the exact solution and proved these estimates which are crucial to our analysis as well as the analysis of [2]. The solution U_ϵ of the singular system is constructed in Theorem 7.1 of [11], with the use of the following iteration scheme:

$$(1.37) \quad \begin{aligned} a) \quad & \partial_{x_d} U_\epsilon^{n+1} + \sum_{j=0}^{d-1} \tilde{A}_j(\epsilon U_\epsilon^n) \left(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\epsilon} \right) U_\epsilon^{n+1} = F(\epsilon U_\epsilon^n) U_\epsilon^n, \\ b) \quad & B(\epsilon U_\epsilon^n)(U_\epsilon^{n+1})|_{x_d=0} = G(x', \theta_0), \\ c) \quad & U_\epsilon^{n+1} = 0 \text{ in } t < 0. \end{aligned}$$

Once the U_ϵ^n are obtained, U_ϵ is found by taking the limit as $n \rightarrow \infty$ of the U_ϵ^n .

The stronger result we will show in order to get Theorem 1.16 is the following:

⁴The profile equations are not solved exactly. Description of the sense in which error terms for the profile equations are to be small can be found in the error analysis. Essentially, the error must satisfy the hypotheses of Proposition 2.40.

Theorem 1.17. *Define*

$$(1.38) \quad \mathcal{U}_\epsilon^0(x, \theta_0) = \mathcal{V}^0\left(x, \theta_0 + \underline{\omega}_1 \frac{x_d}{\epsilon}, \dots, \theta_0 + \underline{\omega}_M \frac{x_d}{\epsilon}\right)$$

where \mathcal{V}^0 is as in (1.30). If $G(x', \theta_0) \in H_T^{s+1}$ for sufficiently large s , then for the exact solution U_ϵ of (1.35), we have

$$(1.39) \quad |U_\epsilon(x, \theta_0) - \mathcal{U}_\epsilon^0(x, \theta_0)|_{L^\infty(x_d, H^{s-1}(x', \theta_0))} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

To handle the supremum norms taken over $x_d \in [0, \infty)$ and to accommodate the appearances of $\frac{x_d}{\epsilon}$ in the M periodic arguments of (1.38), we introduce the placeholder $\xi_d = \frac{x_d}{\epsilon}$. Thus, we consider a function

$$(1.40) \quad \mathcal{U}^0(x, \theta_0, \xi_d) := \mathcal{V}^0(x, \theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d) = \underline{v}(x) + \sum_{m=1}^M \sum_{k=1}^{\mu_m} \sigma_{m,k}(x, \theta_0 + \underline{\omega}_m \xi_d) r_{m,k},$$

which we may also write as

$$(1.41) \quad \mathcal{U}^0(x, \theta_0, \xi_d) = \underline{v}(x) + \sum_{m=1}^M \sum_{k=1}^{\mu_m} \psi_{m,k}(x, \theta_0, \xi_d) r_{m,k},$$

where profiles $\psi_{m,k}(x, \theta_0, \xi_d)$ are given in terms of the *periodic profiles* $\sigma_{m,k}(x, \theta_m)$ by

$$(1.42) \quad \psi_{m,k}(x, \theta_0, \xi_d) := \sigma_{m,k}(x, \theta_0 + \underline{\omega}_m \xi_d).$$

As a result, assertions such as Lemma 2.42 and Proposition 2.40, which concern substitutions of the form

$$(1.43) \quad \mathcal{U}_\epsilon^0(x, \theta_0) = \mathcal{U}^0(x, \theta_0, \xi_d)|_{\xi_d = \frac{x_d}{\epsilon}},$$

play an important role in the error analysis.

Remark 1.18. When $\underline{\omega}_m$ is real, since $\sigma_{m,k}(x, \theta_m)$ is periodic in θ_m , the profile $\psi_{m,k}(x, \theta_0, \xi_d)$ defined in (1.42) is almost-periodic in (θ_0, ξ_d) . Furthermore, if β is in the *hyperbolic region* defined in Remark 1.10(iii), all the $\underline{\omega}_m$ are real, and so in that case $\mathcal{U}^0(x, \theta_0, \xi_d)$ is almost-periodic in (θ_0, ξ_d) . In the case that some of the $\underline{\omega}_m$ are nonreal, in order to make sense of the substitutions (1.42) we must first extend the $\sigma_{m,k}(x, \theta_m)$ to be defined for θ_m in the complex plane. This process is detailed in Section 1.2.1.

1.2 Role of nonreal phases and the resulting boundary layers

We now point out a key feature distinguishing our study from that in [2]. In [2] it is assumed that β belongs to the *hyperbolic region*. This is equivalent to requiring that both (i) β is a *regular boundary frequency* (i.e. β satisfies Assumption 1.7) and (ii) all the eigenvalues $\underline{\omega}_m$ of the matrix $\frac{1}{i}\mathcal{A}(\beta)$ are *real*. In our study, we allow for any regular boundary frequency β and must thus handle cases in which some of the $\underline{\omega}_m$ are nonreal. Hence, functions of complex variables must be considered in, for example, (1.26), (1.30), (1.38), and (1.40).

1.2.1 Hyperbolic and elliptic profiles, and the emergence of the elliptic boundary layer

For each m such that $\underline{\omega}_m$ is real, we call the $\sigma_{m,k}(x, \theta_m)$, $k = 1, \dots, \mu_m$, *hyperbolic profiles*, and we note that in order to make the substitution $\theta_m = \theta_0 + \underline{\omega}_m \xi_d$, such as those made in (1.40), a hyperbolic profile needs only to be defined for real θ_m . On the other hand, when $\underline{\omega}_m$ is nonreal, we call the $\sigma_{m,k}(x, \theta_m)$ *elliptic profiles*. For each elliptic profile, as we vary the parameters $\theta_0 \in \mathbb{R}$, $\xi_d \geq 0$, the value $\theta_m = \theta_0 + \underline{\omega}_m \xi_d$ varies throughout one of the half complex planes $\{\text{Im } \theta_m \geq 0\}$, $\{\text{Im } \theta_m \leq 0\}$, depending on the sign of $\text{Im } \underline{\omega}_m$. To make sense of substituting $\theta_m = \theta_0 + \underline{\omega}_m \xi_d$ into the argument of an elliptic profile, we define the profile first for real θ_m and then holomorphically extend the θ_m -dependence into the appropriate half complex plane. To do so, we make use of the Fourier expansions of the profiles.

Consider a profile $\sigma_{m,k}(x, \theta_m)$, periodic in θ_m with mean 0, and its expansion of the form

$$(1.44) \quad \sigma_{m,k}(x, \theta_m) = \sum_{j \in \mathbb{Z} \setminus 0} a_j(x) e^{ij\theta_m}.$$

For the moment we proceed formally, assuming that one can evaluate the above expression at $\theta_m = \theta_0 + \underline{\omega}_m \xi_d$ by substituting $\theta_m = \theta_0 + \underline{\omega}_m \xi_d$ in the argument of each exponential in the expansion for $\sigma_{m,k}(x, \theta_m)$ and yield a convergent expansion for the profile $\psi_{m,k}(x, \theta_0, \xi_d)$ as defined in (1.42). That is, given (1.44), we also have

$$(1.45) \quad \psi_{m,k}(x, \theta_0, \xi_d) = \sum_{j \in \mathbb{Z} \setminus 0} a_j(x) e^{ij(\theta_0 + \underline{\omega}_m \xi_d)}.$$

We remark that, certainly, we have yet to make sense of this sum in the case that $\underline{\omega}_m \in \mathbb{C} \setminus \mathbb{R}$, but also that the space of convergence must be clarified when $\underline{\omega}_m \in \mathbb{R}$. The convergence is made rigorous for both cases with Proposition 2.39. Let us rewrite such an expansion in the following form:

$$(1.46) \quad \psi_{m,k}(x, \theta_0, \xi_d) = \sigma_{m,k}(x, \theta_0 + \underline{\omega}_m \xi_d) = \sum_{j \in \mathbb{Z} \setminus 0} a_j(x) e^{ij\theta_0} e^{ij\operatorname{Re}(\underline{\omega}_m)\xi_d} e^{-j\operatorname{Im}(\underline{\omega}_m)\xi_d}.$$

Suppose $\sigma_{m,k}(x, \theta_m)$ is elliptic. We first consider the case with $\operatorname{Im} \underline{\omega}_m > 0$. Then, the terms in the sum in (1.46) with $j < 0$ grow exponentially with ξ_d . It is easy to check that such terms result in an unsatisfactory candidate for our approximate solution $u_\epsilon^a(x)$, as defined in (1.26), which is nonphysical in the sense that it blows up in L^∞ as $\epsilon \rightarrow 0$. This leads us to construct $\sigma_{m,k}(x, \theta_m)$ such that $a_j(x) = 0$ for $j < 0$, and so an elliptic profile with $\operatorname{Im} \underline{\omega}_m > 0$ is to have in fact the following expansion rather than the one in (1.44):

$$(1.47) \quad \sigma_{m,k}(x, \theta_m) = \sum_{j \in \mathbb{Z}^+ \setminus 0} a_j(x) e^{ij\theta_m}, \quad \text{for } \operatorname{Im} \underline{\omega}_m > 0.$$

Provided the above is a Fourier series in real θ_m converging in $H^{\frac{d}{2}+3}(x, \theta_m)$, one can show that it holomorphically extends into $\{\operatorname{Im} \theta_m \geq 0\}$. Now, for the corresponding profile $\psi_{m,k}(x, \theta_0, \xi_d) = \sigma_{m,k}(x, \theta_0 + \underline{\omega}_m \xi_d)$, we take (1.45) and (1.46), apply the fact that $a_j(x) = 0$ for $j < 0$, and get

$$(1.48) \quad \psi_{m,k}(x, \theta_0, \xi_d) = \sum_{j \in \mathbb{Z}^+ \setminus 0} a_j(x) e^{ij(\theta_0 + \underline{\omega}_m \xi_d)} = \sum_{j \in \mathbb{Z}^+ \setminus 0} a_j(x) e^{ij\theta_0} e^{ij\operatorname{Re}(\underline{\omega}_m)\xi_d} e^{-j\operatorname{Im}(\underline{\omega}_m)\xi_d}, \quad \text{for } \operatorname{Im} \underline{\omega}_m > 0.$$

As a result, the profile $\psi_{m,k}(x, \theta_0, \xi_d)$ must decay exponentially in ξ_d .

In the case that $\operatorname{Im} \underline{\omega}_m < 0$, similar considerations lead us to construct a corresponding elliptic profile of the form

$$(1.49) \quad \sigma_{m,k}(x, \theta_m) = \sum_{j \in \mathbb{Z}^- \setminus 0} a_j(x) e^{ij\theta_m}, \quad \text{for } \operatorname{Im} \underline{\omega}_m < 0,$$

a sum which holomorphically extends into $\{\operatorname{Im} \theta_m \leq 0\}$. This kind of elliptic profile also results in $\psi_{m,k}(x, \theta_0, \xi_d) = \sigma_{m,k}(x, \theta_0 + \underline{\omega}_m \xi_d)$ which decays exponentially as ξ_d increases.

For a hyperbolic profile $\sigma_{m,k}(x, \theta_m)$, which has $\operatorname{Im} \underline{\omega}_m = 0$, we do not make such restrictions on its coefficients $a_j(x)$, so we describe it with (1.44) and $\psi_{m,k}(x, \theta_0, \xi_d)$ with the expansions (1.45) and (1.46).

With the following remark, we summarize the different forms of expansions for profiles.

Remark 1.19. For $Z_m = \{n \in \mathbb{Z} : n\operatorname{Im} \underline{\omega}_m \geq 0\}$, i.e.

$$(1.50) \quad Z_m = \begin{cases} \mathbb{Z} & \text{if } \underline{\omega}_m \in \mathbb{R}, \\ \mathbb{Z}^+ & \text{if } \operatorname{Im} \underline{\omega}_m > 0, \\ \mathbb{Z}^- & \text{if } \operatorname{Im} \underline{\omega}_m < 0, \end{cases}$$

each periodic profile has an expansion of the form

$$(1.51) \quad \sigma_{m,k}(x, \theta_m) = \sum_{j \in \mathbb{Z}_m \setminus 0} a_j(x) e^{ij\theta_m},$$

and each profile $\psi_{m,k}(x, \theta_0, \xi_d)$ has

$$(1.52) \quad \psi_{m,k}(x, \theta_0, \xi_d) = \sum_{j \in \mathbb{Z}_m \setminus 0} a_j(x) e^{ij(\theta_0 + \underline{\omega}_m \xi_d)} = \sum_{j \in \mathbb{Z}_m \setminus 0} a_j(x) e^{ij\theta_0} e^{ij\operatorname{Re}(\underline{\omega}_m)\xi_d} e^{-j\operatorname{Im}(\underline{\omega}_m)\xi_d}.$$

Since the elliptic profiles $\sigma_{m,k}(x, \theta_m)$ result in $\psi_{m,k}(x, \theta_0, \xi_d)$ which decay in ξ_d , upon replacing the placeholder ξ_d with $\frac{x_d}{\epsilon}$ we see they contribute to a boundary layer with rapid exponential decay in x_d for small ϵ , i.e. the *elliptic boundary layer*, appearing in the approximate solution described by (1.26).

It appears that this paper is the first to rigorously justify leading order expansions involving multiple real and nonreal phase functions $\phi_m(x)$, and thus both hyperbolic and elliptic profiles, in quasilinear hyperbolic boundary problems.

1.2.2 Loss of almost-periodicity and construction of the corrector

In [2], the authors constructed leading order expansions similar to ours, though aspects of the analysis do not work with nonreal phases. As discussed in Remark 1.18, in the case studied in [2], all the $\underline{\omega}_m$ are real, a condition which generally yields $\mathcal{U}^0(x, \theta_0, \xi_d)$ which is almost-periodic in (θ_0, ξ_d) . Thus, in that study, rather than having to define a kind of convergence of infinite sums of the form (1.46), since the functions of interest were all almost-periodic, it sufficed to work in the space \mathcal{P}_T^s , defined by taking the closure of the set of finite trigonometric polynomials in \mathcal{E}_T^s , defined by

$$(1.53) \quad \mathcal{E}_T^s = \{\mathcal{U}(x, \theta_0, \xi_d) : \sup_{\xi_d \geq 0} |\mathcal{U}(\cdot, \cdot, \xi_d)|_{E_T^s} < \infty\},$$

where

$$(1.54) \quad E_T^s = C(x_d, H_T^s(x', \theta_0)) \cap L^2(x_d, H_T^{s+1}(x', \theta_0)).$$

However, in the case that some of the $\underline{\omega}_m$ are nonreal, one loses the almost-periodicity of $\mathcal{U}^0(x, \theta_0, \xi_d)$. The introduction of exponential decay in ξ_d of the corresponding $\psi_{m,k}(x, \theta_0, \xi_d)$ presents serious obstacles to straightforwardly adopting the approach of [2]. Many estimates proved in [2] appear to have no analogue in our study.

As an example, we consider the construction of a projector used in [2], which we denote by \mathbb{P} . The operator \mathbb{P} is first defined on finite polynomials of the form

$$(1.55) \quad H(x, \theta_0, \xi_d) = \sum_{\kappa = (\kappa_0, \kappa_d) \in \mathbb{Z} \times \mathbb{R}} H_\kappa(x) e^{i\kappa_0 \theta_0 + i\kappa_d \xi_d},$$

and is constructed such that $(\mathbb{P}H)(x, \theta_0, \xi_d) = 0$ if and only if there exists a solution \mathcal{U} of

$$(1.56) \quad \mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d})\mathcal{U} = H(x, \theta_0, \xi_d),$$

where $\mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d}) = \mathcal{A}(\beta)\partial_{\theta_0} + \partial_{\xi_d}$. Now suppose that, in an attempt to account for the existence of nonreal eigenvalues $\underline{\omega}_m$ of $\frac{1}{i}\mathcal{A}(\beta)$, we take H to be a finite polynomial of the form

$$(1.57) \quad H(x, \theta_0, \xi_d) = \sum_{\kappa = (\kappa_0, \kappa_d) \in \mathbb{Z} \times \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}} H_\kappa(x) e^{i\kappa_0 \theta_0 + i\kappa_d \xi_d},$$

noting the decay of H in ξ_d . Given the existence of more than two nonreal eigenvalues of $\frac{1}{i}\mathcal{A}(\beta)$, we found that a linear operator \mathbb{P} satisfying the same for such H in any sensible function space⁵ *typically is unbounded*.

⁵The nonlinearity of (1.1) forces us to consider the action of $\mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d})$ (and thus \mathbb{P}) on vectors such as $re^{i\kappa_d \xi_d} \in \mathbb{R}^N$ for all κ_d in some set closed under \mathbb{Z} -linear combinations; in the event that more than two elliptic modes exist which are pairwise independent over \mathbb{Q} , it follows there exists a sequence $re^{i\kappa_d^n \xi_d}$ in the kernel of \mathbb{P} converging in L^∞ to an element not in the kernel.

On the other hand, the assumptions in [2] lead to having no such eigenvalues, and in that case the projector \mathbb{P} can indeed be continuously extended to \mathcal{P}_T^s . In fact, $\mathbb{P} : \mathcal{P}_T^s \rightarrow \mathcal{P}_T^s$ is used in [2] to obtain a corrector term for the expansion which is crucial to the error analysis of [2]. We also need a corrector but we construct it in a different way.

To obtain a satisfactory corrector term, we found there was no need to define a projector on the space of almost-periodic profiles, such as \mathbb{P} . In fact, we were able to dispense entirely with almost-periodic formulation of the profile equations, such as the one used in [2]. Instead, it was sufficient to solve profile equations and use projectors in just the space for M -periodic functions, which contains the $\mathcal{V}^0(x, \theta_1, \dots, \theta_M)$ and $\mathcal{V}^1(x, \theta_1, \dots, \theta_M)$ featured in (1.30) and (1.31), namely $H^s(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M)$, and therein we found we could naturally phrase much of the error analysis and construct a useful corrector term. In much of the error analysis, in place of the almost-periodic profiles with expansions in terms similar to (1.55), we deal with corresponding Fourier expansions in terms $e^{i\alpha \cdot \theta}$, $\alpha \in \mathbb{Z}^M$ (see Definition 2.1,) $\theta = (\theta_1, \dots, \theta_M)$, holomorphically extended in θ to a selected subset \mathbf{C}^M of \mathbb{C}^M (see Remark 2.3,) circumventing the use of almost-periodicity of [2]. The key observation was that for the construction of an appropriate corrector in $H^s(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M)$, hyperbolic objects as well as elliptic objects could be handled simultaneously, as the terms $e^{i\alpha \cdot \theta}$ behave no worse for $\theta \in \mathbf{C}^M \subset \mathbb{C}^M$ than they do at $\theta \in \mathbb{R}^M$.

Interestingly, working with projectors only in $H^s(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M)$ to obtain the corrector appeared to be significantly simpler than attempting to do so in a space similar to the one used in [2]. For example, the continuity of our projectors is almost immediate,⁶ while even in the case treated in [2], the proof of continuity of the projector \mathbb{P} takes some effort. The trade-off for the gained simplicity is that we must describe convergence of expansions such as (1.46) carefully with Proposition 2.39, instead of just using a space which is a closure of trigonometric polynomials and assuming all the functions needed are almost-periodic. Finally, while we do as much of the error analysis in $H^s(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M)$ as we can, we are forced to introduce the substitutions $\theta = (\theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d)$ and $\xi_d = \frac{x_d}{\epsilon}$ to use [11] for the error analysis, with Proposition 2.43.

1.2.3 Complex transport equations and resonances

In contrast with the real transport equations of the profile equations for the hyperbolic profiles, the complex transport equations for the elliptic profiles are generally not solvable. In [10], a Taylor expansion in x_d is developed to approximately solve the corresponding complex transport equations of the profile equations, but using the usual Taylor error bounds would require us to consider C^k regularity in place of H^s regularity, presenting difficulties to our strategy for leading-order justification. Interestingly, however, in our proof of Theorem 2.41 we were able to show that merely requiring the complex transport equations of the profile equations to hold to first order at $x_d = 0$ is sufficient for achieving L^∞ convergence. The error from the complex transport equations belongs to a class of functions in $H_T^s(\overline{\mathbb{R}}_+^{d+1}, \mathbb{T}^M)$ which are *elliptically polarized*.⁷ This error term is thus handled by an application of Proposition 2.40, which clarifies the sense in which such functions are small if they are zero at the boundary $\{x_d = 0\}$.

Another unusual feature involving elliptic phases in the analysis is the handling of complex resonances, which result in couplings amongst the transport equations for the profiles. Careful examination of these resonances revealed that we could first work strictly within a subsystem of real transport equations *not dependent on the elliptic profiles* and solve for the hyperbolic profiles. On the other hand, resonances between hyperbolic and elliptic phases do affect the complex transport equations for the elliptic profiles.

While, as in [2], hyperbolic profiles $\sigma_{m,k}$ are obtained as limits of $\sigma_{m,k}^n$ via an iteration scheme, there is no need to develop such a scheme to obtain the elliptic profiles. We construct satisfactory elliptic $\sigma_{m,k}$ by using finite propagation and regularity properties resulting from solving wave equations where time is thought of as the x_d variable rather than the t variable and using a trick of taking a frame of reference moving at the speed of propagation to ensure the $\sigma_{m,k}$ are supported in $t \geq 0$. The conditions which the elliptic profiles must satisfy at $x_d = 0$ are straightforwardly read off from linear relations with the hyperbolic profiles at $x_d = 0$ and G . Inserting the values of the profiles at $x_d = 0$ into the complex transport equation evaluated on the boundary and requiring this to hold determines the values of the x_d -derivatives of the elliptic profiles on

⁶See \mathbb{E} and \mathbb{E}^b appearing in Definition 2.6 and Remark 2.7.

⁷These functions satisfy $\mathbb{E}_e \mathcal{V} = \mathcal{V}$, where \mathbb{E}_e is the *elliptic projector* defined in (2.23).

the boundary⁸. However, we still develop a sequence of elliptic ‘iterates’ $\sigma_{m,k}^n$; we emphasize that while they do converge to the elliptic $\sigma_{m,k}$, these elliptic iterates are not used in the construction of the elliptic $\sigma_{m,k}$. The elliptic iterates are useful because they are convenient for the error analysis, allowing us to perform much of the analysis on the elliptic and hyperbolic parts simultaneously and in as uniform a way as possible.

1.3 Use of simultaneous Picard iteration in the error analysis

In the proof of Theorem 7.1 of [11], for some $T_0 > 0$, the iteration scheme for the singular system, (1.37), is used to produce $U_\epsilon(x, \theta_0)$ and iterates $U_\epsilon^n(x, \theta_0)$ bounded in $E_{T_0}^s$ uniformly with respect to n and ϵ and which satisfy

$$(1.58) \quad \lim_{n \rightarrow \infty} U_\epsilon^n = U_\epsilon \text{ in } E_{T_0}^{s-1} \text{ uniformly with respect to } \epsilon \in (0, \epsilon_0],$$

where U_ϵ solves the singular system (1.36).

In Section 2.3, Proposition 2.30 of Section 2.4, and Definition 2.31 we construct a function $\mathcal{V}^0(x, \theta) \in H^s(\overline{\mathbb{R}_+^{d+1}} \times \mathbb{T}^M)$ which approximately satisfies the profile equations, and iterates $\mathcal{V}^{0,n}(x, \theta)$ approximately satisfying the equations of a corresponding iteration scheme. The iterates $\mathcal{V}^{0,n}$ are bounded in $\mathbb{H}_{T_0}^{s+1}$ uniformly with respect to n and satisfy

$$(1.59) \quad \lim_{n \rightarrow \infty} \mathcal{V}^{0,n} = \mathcal{V}^0 \text{ in } \mathbb{H}_{T_0}^s.$$

Again, we make the substitution seen in (1.40), plugging in $\theta = (\theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d)$, to get

$$(1.60) \quad \mathcal{U}^{0,n}(x, \theta_0, \xi_d) := \mathcal{V}^{0,n}(x, \theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d),$$

followed by $\xi_d = \frac{x_d}{\epsilon}$, yielding

$$(1.61) \quad \mathcal{U}_\epsilon^{0,n}(x, \theta_0) := \mathcal{V}^{0,n}\left(x, \theta_0 + \underline{\omega}_1 \frac{x_d}{\epsilon}, \dots, \theta_0 + \underline{\omega}_M \frac{x_d}{\epsilon}\right).$$

Thus, by using (1.59) and applying the estimates given by Proposition 2.39 and Lemma 2.42, we get that

$$(1.62) \quad \lim_{n \rightarrow \infty} \mathcal{U}_\epsilon^{0,n} = \mathcal{U}_\epsilon^0 \text{ in } E_{T_0}^{s-1} \text{ uniformly with respect to } \epsilon \in (0, \epsilon_0],$$

where \mathcal{U}_ϵ^0 is as in Theorem 1.17. Therefore, to conclude $\lim_{\epsilon \rightarrow 0} \mathcal{U}_\epsilon^0(x, \theta_0) - U_\epsilon(x, \theta_0) = 0$ in $E_{T_0}^{s-1}$ and finish the proof of Theorem 1.17, it is sufficient to show

$$(1.63) \quad \lim_{\epsilon \rightarrow 0} |\mathcal{U}_\epsilon^{0,n} - U_\epsilon^n|_{E_{T_0}^{s-1}} = 0 \text{ for all } n.$$

Indeed, (1.63) is proved in Section 2.6 by induction on n . One might try to prove the statement in this way by applying the estimate for the linearized singular system of Proposition 2.43 to $(\mathcal{U}_\epsilon^{0,n+1} - U_\epsilon^{n+1})$. The problem with this is that if we take $\mathbb{A}(\epsilon U_\epsilon^n)$, which we define to be the operator appearing in the left hand side of the equation for U_ϵ^{n+1} , i.e. (1.37)(a), and apply it to the difference $(\mathcal{U}_\epsilon^{0,n+1} - U_\epsilon^{n+1})$, the resulting quantity does not necessarily converge to zero as $\epsilon \rightarrow 0$. To illustrate this point, consider the relation of \mathcal{U}_ϵ^0 to \mathcal{V}^0 . Recall that \mathcal{V}^0 is constructed⁹ to satisfy (1.29) so that in (1.27) the order $\frac{1}{\epsilon}$ terms are annihilated, noting that the terms of orders $\epsilon, \epsilon^2, \dots$ shrink to zero with ϵ , but observe that we have yet to handle the $O(1)$ terms. Similarly, plugging in $\mathcal{U}_\epsilon^{0,n+1}$, an approximation for \mathcal{U}_ϵ^0 , in place of U_ϵ^{n+1} in the equation for U_ϵ^{n+1} results in an $O(1)$ error.

This could be handled if a corrector for $\mathcal{V}^{0,n+1}$ could be constructed, say \mathcal{V}^1 , analogous to the \mathcal{V}^1 appearing in (1.31), which resulted in the elimination of the $O(1)$ terms upon the replacement of $\mathcal{V}^{0,n+1}$ by $\mathcal{V}^{0,n+1} + \epsilon \mathcal{V}^1$ in a similar equation. Then one obtains a corrector for $\mathcal{U}_\epsilon^{0,n+1}$, the function $\mathcal{U}_\epsilon^1(x, \theta_0)$, from $\mathcal{V}^1(x, \theta)$ by plugging in $\theta = (\theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d)$ and $\xi_d = \frac{x_d}{\epsilon}$ as before. In fact, applying $\mathbb{A}(\epsilon \mathcal{U}_\epsilon^{0,n})$ to $(\mathcal{U}_\epsilon^{0,n+1} + \epsilon \mathcal{U}_\epsilon^1 - U_\epsilon^{n+1})$ results in an error which converges to zero as $\epsilon \rightarrow 0$ given the existence of suitable

⁸For further discussion, see Section 2.4.

⁹Strictly speaking, details on the construction of \mathcal{V}^0 come later in the paper with the solution of the profile equations. Instead, this refers to imposing the condition that \mathcal{V}^0 has the form (1.30).

\mathcal{V}^1 . After an application of the estimate from [11], (1.63) can then be obtained, assuming \mathcal{U}_ϵ^1 is bounded by $C\epsilon$ in an appropriate norm.

However, there are two major obstacles to constructing such a corrector \mathcal{V}^1 for $\mathcal{V}^{0,n+1}$. The first is that each of our iterates $\mathcal{V}^{0,n+1}$ only *approximately* solves the corresponding iterate equation of the profile equations. This is forced on us since our profile equations include complex transport equations which are not generally exactly solvable. On the other hand, as mentioned in Remark 1.15, typically in geometric optics, the solvability conditions on an ansatz such as \mathcal{V}^0 which imply \mathcal{V}^0 has a corrector are satisfied by *exactly* solving profile equations; similarly, in [2], solvability is achieved through exact solutions of the iterated profile equations. Instead, however, we found it was sufficient to solve each of the iterated profile equations up to an error which is zero at the boundary $\{x_d = 0\}$ and ‘purely elliptic’ in the sense that it depends only on components θ_m which are the arguments of elliptic profiles and is consequently *elliptically polarized*; these conditions on the error are described precisely with the hypotheses of Proposition 2.40. Essentially, we form new solvability conditions which are the iterated profile equations modified by including the error terms.

The second issue is that, since we do not make a small-divisor assumption, we can only guarantee solvability if we are working with *finite* trigonometric polynomials¹⁰, as opposed to general elements of $H_{T_0}^s$ with infinitely many nonzero Fourier coefficients. To resolve this, we approximate each $\mathcal{V}^{0,n+1}$ by a finite trigonometric polynomial $\mathcal{V}_p^{0,n+1}$, construct a corresponding corrector \mathcal{V}_p^1 , and use Proposition 2.39 and Lemma 2.42 to work with finite sums $\mathcal{U}_p^{0,n+1}$ and $\mathcal{U}_{p,\epsilon}^{0,n+1}$ approximating $\mathcal{U}^{0,n+1}$ and $\mathcal{U}_\epsilon^{0,n+1}$. Using these approximations, we are finally able to conclude (1.63).

2 Profile equations: formulation with periodic profiles

Definition 2.1. (i) We define $Z^M \subset \mathbb{Z}^M$ by

$$(2.1) \quad Z^M := \{\alpha = (\alpha_i)_{i=1}^M : \alpha_i \in Z_i\},$$

where Z_i is defined by

$$(2.2) \quad Z_i := \begin{cases} \mathbb{Z} & \text{for } i \in \mathcal{I} \cup \mathcal{O}, \\ \mathbb{Z}^+ & \text{for } i \in \mathcal{P}, \\ \mathbb{Z}^- & \text{for } i \in \mathcal{N}, \end{cases}$$

where \mathcal{I} , \mathcal{O} , \mathcal{P} , and \mathcal{N} are as defined in the comments following Definition 1.12. We also have the equivalent formulation $Z_i = \{n \in \mathbb{Z} : n \operatorname{Im} \underline{\omega}_i \geq 0\}$.

(ii) We also define

$$(2.3) \quad Z^{M;k} := \{\alpha \in Z^M : \text{at most } k \text{ components of } \alpha \text{ are nonzero}\}.$$

Definition 2.2. For $k = 1, 2$, we define the following spaces:

$$(2.4) \quad H^{s;k}(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M) = \left\{ \mathcal{V}(x, \theta) \in H^s(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M) : \mathcal{V}(x, \theta) = \sum_{\alpha \in Z^{M;k}} V_\alpha(x) e^{i\alpha \cdot \theta} \right\}.$$

For $s > (d+1+2)/2$, it is clear that multiplication defines a continuous map

$$(2.5) \quad H^{s;1}(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M) \times H^{s;1}(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M) \rightarrow H^{s;2}(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M).$$

Remark 2.3. We define

$$(2.6) \quad \mathbf{C}^M := \mathbf{C}_1 \times \mathbf{C}_2 \times \cdots \times \mathbf{C}_M,$$

¹⁰For details on solvability, see Proposition 2.9.

where \mathbf{C}_i is defined by

$$(2.7) \quad \mathbf{C}_i := \begin{cases} \mathbb{R} & \text{for } i \in \mathcal{I} \cup \mathcal{O}, \\ \{\operatorname{Im} z \geq 0\} & \text{for } i \in \mathcal{P}, \\ \{\operatorname{Im} z \leq 0\} & \text{for } i \in \mathcal{N}. \end{cases}$$

For $\mathcal{V} \in H^{s;2}(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M)$ where $s > \frac{d}{2} + 3$, one can show that $\operatorname{spec} \mathcal{V} \subset Z^{M;2}$ implies \mathcal{V} extends holomorphically in θ to the interior of \mathbf{C}^M . In particular, this uses the fact that then $\operatorname{Im}(\alpha_i \theta_i) \geq 0$ for $i \in \mathcal{P} \cup \mathcal{N}$. This allows us to make sense of

$$(2.8) \quad \mathcal{U}(x, \theta_0, \xi_d) := \mathcal{V}(x, \theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d)$$

for $\theta_0 \in \mathbb{T}$, $\xi_d \geq 0$.

2.1 The periodic ansatz and its corrector.

Our periodic ansatz will have the form

$$(2.9) \quad \mathcal{V}^0(x, \theta) = \underline{v}(x) + \sum_{m=1}^M \sum_{k=1}^{\mu_m} \sigma_{m,k}(x, \theta_m) r_{m,k},$$

a function in $H^{s;1}(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T}^M)$, where s is to be specified, and which is holomorphic in $\theta \in \mathbf{C}^M \subset \mathbb{C}^M$, in particular. Using the notation

$$(2.10) \quad \theta(\theta_0, \xi_d) = (\theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d),$$

a calculation shows

$$(2.11) \quad u_\epsilon^a(x) := \mathcal{U}^0(x, \theta_0, \xi_d)|_{\theta_0 = \frac{\phi_0}{\epsilon}, \xi_d = \frac{x_d}{\epsilon}} = \mathcal{V}^0(x, \theta(\theta_0, \xi_d))|_{\theta_0 = \frac{\phi_0}{\epsilon}, \xi_d = \frac{x_d}{\epsilon}}$$

results in the vanishing of the terms of order $\frac{1}{\epsilon}$ when plugging u_ϵ^a into $P(\epsilon u_\epsilon, \partial_x) u_\epsilon$ in (1.2)(a). If we plug in a corrected approximate solution

$$(2.12) \quad u_\epsilon^c(x) := (\mathcal{U}^0(x, \theta_0, \xi_d) + \epsilon \mathcal{U}^1(x, \theta_0, \xi_d))|_{\theta_0 = \frac{\phi_0}{\epsilon}, \xi_d = \frac{x_d}{\epsilon}} = (\mathcal{V}^0(x, \theta(\theta_0, \xi_d)) + \epsilon \mathcal{V}^1(x, \theta(\theta_0, \xi_d)))|_{\theta_0 = \frac{\phi_0}{\epsilon}, \xi_d = \frac{x_d}{\epsilon}}$$

where we do not necessarily have \mathcal{V}^1 of the form (2.9), but require $\mathcal{V}^1(x, \theta) \in H^{s;2}(x, \theta)$, then the terms of order ϵ^0 cancel out if and only if

$$(2.13) \quad \mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U}^1 + \tilde{L}(\partial_x) \mathcal{U}^0 + \mathcal{M}'(\mathcal{U}^0) \partial_{\theta_0} \mathcal{U}^0 = F(0) \mathcal{U}^0,$$

using the notation

$$(2.14) \quad \mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d}) := \mathcal{A}(\beta) \partial_{\theta_0} + \partial_{\xi_d},$$

$$(2.15) \quad \mathcal{M}'(\mathcal{U}) := \sum_{j=0}^{d-1} \partial_u \tilde{A}_j(0) \mathcal{U} \beta_j.$$

A sufficient condition for (2.13) is

$$(2.16) \quad \mathcal{L}(\partial_\theta) \mathcal{V}^1 + \tilde{L}(\partial_x) \mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0) \partial_\theta \mathcal{V}^0 = F(0) \mathcal{V}^0,$$

where

$$(2.17) \quad \mathcal{L}(\partial_\theta) = \sum_{m=1}^M \tilde{L}(d\phi_m) \partial_{\theta_m}$$

and

$$(2.18) \quad \mathcal{M}(\mathcal{V})\partial_\theta := \sum_{m=1}^M \sum_{j=0}^{d-1} \beta_j \partial_u \tilde{A}_j(0) \mathcal{V} \partial_{\theta_m}.$$

Observe that the operator $\mathcal{L}(\partial_\theta)$ is singular, so that one cannot simply invert it to solve for \mathcal{V}^1 in (2.16). One approach is to ensure the existence of a solution \mathcal{V}^1 by imposing the following condition on \mathcal{V}^0 :

$$(2.19) \quad \tilde{L}(\partial_x) \mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0) \partial_\theta \mathcal{V}^0 - F(0) \mathcal{V}^0 \in \text{Im } \mathcal{L}(\partial_\theta),$$

a condition which turns out to be equivalent to a differential equation¹¹ in \mathcal{V}^0 .

The condition (2.19) on \mathcal{V}^0 does not clearly determine \underline{v} and the $\sigma_{m,k}$, the parts of \mathcal{V}^0 which remain to be chosen. Strictly speaking, we will not satisfy (2.16) or (2.19). Instead, we replace the differential equation (equivalent to (2.19)) with another one¹² which agrees when we plug in $\theta = \theta(\theta_0, \xi_d)$ and can be decomposed into the system of profile equations (in \underline{v} and the $\sigma_{m,k}$.) This allows us to solve for the profiles and a corrector \mathcal{V}^1 satisfying an equation which agrees with (2.16) on $\theta = \theta(\theta_0, \xi_d)$. Thus, this condition is *also* sufficient for (2.13).

Now we work towards defining the projection operators to appear in these differential equations, \mathbb{E} and \mathbb{E}^b , which will allow us to construct the ansatz \mathcal{V}^0 and its corrector \mathcal{V}^1 .

Definition 2.4. Setting $\phi := (\phi_1, \dots, \phi_M)$, we call $\alpha \in Z^{M;2}$ a characteristic mode provided $\det L(d(\alpha \cdot \phi)) = 0$ and write $\alpha \in \mathcal{C}$. We decompose

$$(2.20) \quad \mathcal{C} = \cup_{m=1}^M \mathcal{C}_m, \text{ where } \mathcal{C}_m = \{\alpha \in Z^{M;2} : \alpha \cdot \phi = n_\alpha \phi_m \text{ for some } n_\alpha \in \mathbb{Z}\}.$$

Remark 2.5. Normally, the above is defined with $\mathbb{Z}^{M;2}$ in place of $Z^{M;2}$, but in preventing our solution from blowing up, we have restricted its spectrum such that only $\alpha \in Z^{M;2}$ are considered.

Now we are ready to define the projectors which will give us the profile equations and the differential equation for the condition (2.19).

Definition 2.6. Let $\mathcal{V} \in H_T^{s+1;2}$. The action of \mathbb{E} on \mathcal{V} is defined by

$$(2.21) \quad \mathbb{E} = \mathbb{E}_0 + \sum_{m=1}^M \mathbb{E}_m, \text{ where } \mathbb{E}_0 \mathcal{V} = V_0 \text{ and } \mathbb{E}_m \mathcal{V} = \sum_{\alpha \in \mathcal{C}_m \setminus 0} P_m V_\alpha(x) e^{in_\alpha \theta_m},$$

where P_m denotes the projection onto $\text{Ker } L(d\phi_m)$, the action of \mathbb{E}^b on \mathcal{V} is defined by

$$(2.22) \quad \mathbb{E}^b = \mathbb{E}_0^b + \sum_{m=1}^M \mathbb{E}_m^b, \text{ where } \mathbb{E}_0^b \mathcal{V} = V_0 \text{ and } \mathbb{E}_m^b \mathcal{V} = \sum_{\alpha \in \mathcal{C}_m \setminus 0} P_m V_\alpha(x) e^{i\alpha \cdot \theta},$$

and we use the notation

$$(2.23) \quad \mathbb{E}_h = \mathbb{E}_0 + \sum_{m \in \mathcal{I} \cup \mathcal{O}} \mathbb{E}_m, \quad \mathbb{E}_e = \sum_{m \in \mathcal{P} \cup \mathcal{N}} \mathbb{E}_m.$$

Remark 2.7. (i) It is shown in [2] that \mathbb{E} is a continuous map in the H^s norm. Here, to complete the statement $\mathbb{E} : H^{s;2} \rightarrow H^{s;1}$, we must also have for each $\alpha \in \mathcal{C}_m$ that $n_\alpha \in Z_m$. This is easily verified from the definitions of Z_m , \mathcal{C}_m , and the n_α . A similar calculation is done in Remark 2.12. (ii) It is not hard to show the continuity of $\mathbb{E}^b : H^{s;2} \rightarrow H^{s;2}$.

Remark 2.8. Denoting evaluation at $\theta = \theta(\theta_0, \xi_d)$ by Φ , observe $\Phi \circ \mathbb{E} = \Phi \circ \mathbb{E}^b$. The projector \mathbb{E} serves as the tool for solving for $\underline{v}(x)$ and the periodic profiles $\sigma_{m,k}(x, \theta_m)$ of our ansatz. The projector \mathbb{E}^b is key to describing solvability (such as the condition (2.19)) and is thus used in the error analysis in solving away an $O(1)$ error output of $\Phi \circ (I - \mathbb{E})$ written in the form $\Phi \circ (I - \mathbb{E}^b)$, so that we can prove Theorem 2.41. That is the purpose which the following proposition serves.

¹¹This differential equation is (2.35).

¹²This is the differential equation (2.37).

Proposition 2.9. *Given $\mathcal{H} \in H_T^{s;2}(x, \theta)$ with finitely many nonzero Fourier coefficients, suppose*

$$(2.24) \quad \mathbb{E}^b \mathcal{H} = 0,$$

Then there exists $\mathcal{V} \in H_T^{s;2}(x, \theta)$ such that

$$(2.25) \quad \mathcal{L}(\partial_\theta) \mathcal{V} = \mathcal{H}.$$

Proof. We write out the series of \mathcal{H} as

$$(2.26) \quad \mathcal{H}(x, \theta) = \sum_{\alpha \in Z^{M;2}} H_\alpha(x) e^{i\alpha \cdot \theta}.$$

Now we search for $V_\alpha(x)$ such that, upon defining

$$(2.27) \quad \mathcal{V}(x, \theta) := \sum_{\alpha \in Z^{M;2}} V_\alpha(x) e^{i\alpha \cdot \theta},$$

one has

$$(2.28) \quad \mathcal{L}(\partial_\theta) \mathcal{V}(x, \theta) = \sum_{m=1}^M \sum_{\alpha \in Z^{M;2}} i\alpha_m \tilde{L}(d\phi_m) V_\alpha(x) e^{i\alpha \cdot \theta} = \sum_{\alpha \in Z^{M;2}} H_\alpha(x) e^{i\alpha \cdot \theta}.$$

This holds if and only if for all $\alpha \in Z^{M;2}$

$$(2.29) \quad i\tilde{L} \left(\sum_m \alpha_m \beta, \alpha \cdot \omega \right) V_\alpha(x) = H_\alpha(x),$$

where $\omega = (\underline{\omega}_1, \dots, \underline{\omega}_M)$. We handle first the case where $\alpha \in \mathcal{C}_j \setminus 0$ for some $j = 1, 2, \dots, M$. Note that

$$(2.30) \quad 0 = \mathbb{E}_j^b \mathcal{H}(x, \theta) = \sum_{\alpha \in \mathcal{C}_j \setminus 0} P_j H_\alpha(x) e^{i\alpha \cdot \theta},$$

which implies for each $\alpha \in \mathcal{C}_j$

$$(2.31) \quad 0 = P_j H_\alpha(x).$$

Recalling from Lemma 1.14 that $\text{Im } A_d^{-1} L(d\phi_j) = \text{Ker } P_j$, we see there exists $W_\alpha(x) \in H_T^s(x)$ such that

$$(2.32) \quad H_\alpha(x) = i\tilde{L}(d\phi_j) W_\alpha(x).$$

Observe then

$$(2.33) \quad H_\alpha(x) = i\tilde{L}(n_\alpha \beta, n_\alpha \underline{\omega}_j) \frac{1}{n_\alpha} W_\alpha(x),$$

$$(2.34) \quad = i\tilde{L} \left(\sum_{m=1}^M \alpha_m \beta, \alpha \cdot \omega \right) \frac{1}{n_\alpha} W_\alpha(x).$$

So we may satisfy (2.29) by defining $V_\alpha(x) = \frac{1}{n_\alpha} W_\alpha(x)$.

For the case $\alpha = 0$, where $\mathbb{E}_0^b \mathcal{H} = H_\alpha = 0$, or for any other α such that $H_\alpha = 0$, we may satisfy (2.29) by taking $V_\alpha = 0$.

For $\alpha \notin \mathcal{C}$ with $H_\alpha \neq 0$, we have $\det L(\sum_m \alpha_m \beta, \alpha \cdot \omega) \neq 0$ by definition of \mathcal{C} , so then (2.29) can be solved directly for $V_\alpha(x)$. Since there are only finitely many nonzero H_α , we find the same holds for the V_α , and so the sum in (2.27) indeed describes an element of $H_T^{s;2}(x, \theta)$. \square

Remark 2.10. (i) In view of Proposition 2.9, in order to construct a corrector \mathcal{V}^1 appropriate for our periodic ansatz \mathcal{V}^0 to satisfy (2.16), it is tempting to require something like

$$(2.35) \quad \mathbb{E}^b(\tilde{L}(\partial_x)\mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0)\partial_\theta\mathcal{V}^0) = \mathbb{E}^b(F(0)\mathcal{V}^0).$$

However, note that to use Proposition 2.9, if the \mathcal{V}^0 we seek has infinitely many nonzero Fourier coefficients, (2.35) alone is insufficient; we must use finite trigonometric polynomial approximations. Furthermore, even if \mathcal{V}^0 is replaced with a finite trigonometric polynomial approximation, there is no clear way to solve (2.35) for \mathcal{V}^0 of the form (2.9). We explain: the components of the projector \mathbb{E}^b are the \mathbb{E}_m^b ; typically, one obtains some profile, say, $\sigma_{m,k}(x, \theta_m)$ in (2.9) by examining the m th component of a projector. However, \mathbb{E}_m^b maps into $H^{s;2}(x, \theta)$, meaning its output generally varies with *more than one* component of θ as opposed to varying with only the θ_m component. Meanwhile, \mathbb{E} has components \mathbb{E}_m mapping into $H^s(x, \theta_m)$, spaces suited to $\sigma_{m,k}(x, \theta_m)$. In fact, \mathcal{V}^0 having the form (2.9) is equivalent to having

$$(2.36) \quad \mathbb{E}\mathcal{V}^0 = \mathcal{V}^0.$$

(ii) With the considerations made in Remark 2.8, it is natural to require instead of (2.35) that

$$(2.37) \quad \mathbb{E}(\tilde{L}(\partial_x)\mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0)\partial_\theta\mathcal{V}^0) = \mathbb{E}(F(0)\mathcal{V}^0).$$

In fact, (2.37) gives the solvability conditions discussed in Remark 1.15 which ensure the existence of a corrected approximate solution $u_\epsilon^c(x)$ resulting in the vanishing of the terms of order $O(1)$ in (1.2)(a). To see this, recall that the $O(1)$ terms vanish if and only if (2.13) holds. If we satisfy (2.37), we have

$$(2.38) \quad (I - \mathbb{E})(\tilde{L}(\partial_x)\mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0)\partial_\theta\mathcal{V}^0 - F(0)\mathcal{V}^0) = \tilde{L}(\partial_x)\mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0)\partial_\theta\mathcal{V}^0 - F(0)\mathcal{V}^0.$$

Once \mathcal{V}^0 is determined by imposing (2.37), with Proposition 2.9 one can solve for a corrector \mathcal{V}^1 in

$$(2.39) \quad \mathcal{L}(\partial_\theta)\mathcal{V}^1 = -(I - \mathbb{E}^b)(\tilde{L}(\partial_x)\mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0)\partial_\theta\mathcal{V}^0 - F(0)\mathcal{V}^0),$$

(if we assume for the moment that we are working with finite trigonometric polynomials.) Although such \mathcal{V}^1 does not necessarily satisfy (2.16), after evaluating at $\theta = \theta(\theta_0, \xi_d)$, which, recall, is denoted by Φ , it follows from (2.38) and (2.39) that

$$(2.40) \quad \Phi(\mathcal{L}(\partial_\theta)\mathcal{V}^1) = -\Phi(\tilde{L}(\partial_x)\mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0)\partial_\theta\mathcal{V}^0 - F(0)\mathcal{V}^0),$$

where we have used the property $\Phi \circ (I - \mathbb{E}^b) = \Phi \circ (I - \mathbb{E})$. In fact, for $\mathcal{U}^1 = \Phi(\mathcal{V}^1)$ and $\mathcal{U}^0 = \Phi(\mathcal{V}^0)$, (2.40) is equivalent to (2.13), giving $u_\epsilon^c(x)$ solving (1.2)(a) to order $O(1)$.

It is from the components of (2.37) that we will obtain the components \underline{v} , $\sigma_{m,k}$, $m = 1, \dots, M$, $k = 1, \dots, \mu_m$, of \mathcal{V}^0 . Now we collect the set of equations which determines $\mathcal{V}^0(x, \theta) \in H^{s;1}(\mathbb{R}_+^{d+1} \times \mathbb{T}^M)$ for s sufficiently large (specified later):

$$(2.41) \quad \begin{aligned} &a) \mathbb{E}\mathcal{V}^0 = \mathcal{V}^0 \\ &b) \mathbb{E}(\tilde{L}(\partial_x)\mathcal{V}^0 + \mathcal{M}(\mathcal{V}^0)\partial_\theta\mathcal{V}^0) = \mathbb{E}(F(0)\mathcal{V}^0) \text{ in } x_d \geq 0 \\ &c) B(0)\mathcal{V}^0(x', 0, \theta_0, \dots, \theta_0) = G(x', \theta_0) \\ &d) \mathcal{V}^0 = 0 \text{ in } t < 0. \end{aligned}$$

In summary, (2.41)(a) gives that $\mathcal{L}(\partial_\theta)\mathcal{V}^0 = 0$, and as a result, in the system for the original solution as a perturbation, this guarantees that (1.2)(a) is solved to order $O(\frac{1}{\epsilon})$. Equation (2.41)(b) represents the solvability conditions discussed in Remark 1.15, guaranteeing the existence of a corrected approximate solution $u_\epsilon^c(x)$ as seen in (1.31) which solves (1.2)(a) to order $O(1)$. Finally, equations (2.41)(c) and (2.41)(d) straightforwardly correspond to (1.2)(b) and (1.2)(c).

Resonances.

The next major task is to rephrase (2.41)(b) in terms of \underline{v} and the $\sigma_{m,k}$ to get the interior profile equations. (2.41)(b) will be broken down into components where \mathbb{E} is replaced by E_0 and \mathbb{E}_p , $p = 1, \dots, M$,

yielding equations for \underline{v} and the $\sigma_{p,l}$, $l = 1, \dots, \mu_p$. The nonlinearity in (2.41)(b) leads to resonances, which are generated in products such as $\sigma_{q,k}(x, \frac{\phi_q}{\epsilon}) \partial_{\theta_r} \sigma_{r,k'}(x, \frac{\phi_r}{\epsilon})$ when there exists a relation of the form

$$(2.42) \quad n_p \phi_p = n_q \phi_q + n_r \phi_r \text{ where } p \in \{1, \dots, M\} \setminus \{q, r\} \text{ and } (p, q, r) \in \mathbb{Z} \times Z_q \times Z_r.$$

This means that ϕ_q oscillations interact with ϕ_r relations to produce ϕ_p oscillations. While such ϕ_p oscillations arise on their own in products such as $\sigma_{q,k}(x, \frac{\phi_q}{\epsilon}) \partial_{\theta_r} \sigma_{r,k'}(x, \frac{\phi_r}{\epsilon})$, not even some θ_p dependence is present in the product $\sigma_{q,k}(x, \theta_q) \partial_{\theta_r} \sigma_{r,k'}(x, \theta_r)$ before we evaluate $\theta_p = \frac{\phi_p}{\epsilon}$, $\theta_q = \frac{\phi_q}{\epsilon}$, $\theta_r = \frac{\phi_r}{\epsilon}$. Meanwhile \mathbb{E}_p accounts for these interactions when it maps such (θ_q, θ_r) oscillations to θ_p oscillations, as it maps $H^{s;2}(x, \theta)$ to $H^s(x, \theta_p)$. When we express $\mathbb{E}_p(2.41)(b)$ solely in terms of functions of (x, θ_p) , it takes some effort to evaluate $\mathbb{E}_p(\mathcal{M}(\mathcal{V}^0) \partial_\theta \mathcal{V}^0)$ in terms of the profiles, in particular to describe these interactions in terms of $\sigma_{q,k}$ and $\sigma_{r,k'}$. This is the purpose of the interaction integrals, (2.57), appearing in the profile equations. To handle these difficulties we introduce some new definitions regarding resonances and interaction integrals.

Definition 2.11. *We say (ϕ_p, ϕ_q, ϕ_r) is an ordered triple of resonant phases and that (ϕ_q, ϕ_r) forms a p -resonance if*

$$(2.43) \quad n_p \phi_p = n_q \phi_q + n_r \phi_r$$

for some $(n_p, n_q, n_r) \in \mathbb{Z} \times Z_q \times Z_r$, each entry nonzero. The triple (ϕ_p, ϕ_q, ϕ_r) is called normalized if we have both (i) $\gcd(n_p, n_q, n_r) = 1$ and (ii) if $p \in \mathcal{I} \cup \mathcal{O}$, then $n_p > 0$.

First, we explicitly treat the case that the only normalized triples of resonant phases are permutations of (ϕ_p, ϕ_q, ϕ_r) corresponding to one of the six rearrangements of (2.43). It follows from the next remark that the sign of n_p is determined, so this normalization uniquely determines n_p, n_q, n_r .

Remark 2.12. For a triple of resonant phases (ϕ_p, ϕ_q, ϕ_r) with (2.43), one has $(n_p, n_q, n_r) \in Z_p \times Z_q \times Z_r$. To see this, taking the imaginary part of $n_p \phi_p = n_q \phi_q + n_r \phi_r$, observe

$$(2.44) \quad n_p \text{Im } \underline{\omega}_p = n_q \text{Im } \underline{\omega}_q + n_r \text{Im } \underline{\omega}_r.$$

Thus, if (ϕ_q, ϕ_r) forms a p -resonance, since $n_q \text{Im } \underline{\omega}_q$ and $n_r \text{Im } \underline{\omega}_r$ are nonnegative, so is $n_p \text{Im } \underline{\omega}_p$, so it follows that $n_p \in Z_p$.

We now make a related observation with the following proposition which will greatly simplify our solution of the interior equations.

Proposition 2.13. *(i) A hyperbolic resonance (i.e. a p -resonance, where $p \in \mathcal{I} \cup \mathcal{O}$) can only be formed by a pair of hyperbolic phases, and (ii) an elliptic resonance can only be formed by a pair including at least one elliptic phase.*

Proof. (i) Suppose for some $p \in \mathcal{I} \cup \mathcal{O}$ that $n_p \phi_p = n_q \phi_q + n_r \phi_r$ with $(n_p, n_q, n_r) \in Z_p \times Z_q \times Z_r$. Considering the imaginary part gives

$$(2.45) \quad 0 = n_q \text{Im } \underline{\omega}_q + n_r \text{Im } \underline{\omega}_r,$$

so recalling $n_i \in Z_i$, $n_i \text{Im } \underline{\omega}_i \geq 0$, we see $\text{Im } \underline{\omega}_q = \text{Im } \underline{\omega}_r = 0$. Hence $q, r \in \mathcal{I} \cup \mathcal{O}$. The proof of (ii) is similar. \square

2.2 The large system for individual profiles.

Recall the decomposition of the projector

$$(2.46) \quad \mathbb{E} = \mathbb{E}_0 + \sum_{m=1}^M \mathbb{E}_m$$

For $m = 1, \dots, M$, we enumerate by $\{\ell_{m,k}, k = 1, \dots, \nu_m\}$ a basis of vectors for the left eigenspace of

$$(2.47) \quad i\mathcal{A}(\beta) = A_d^{-1}(0)(\mathcal{I}I + \sum_{j=1}^{d-1} A_j(0)\underline{\eta}_j)$$

associated to the eigenvalue $-\underline{\omega}_m$, and with the following property:

$$(2.48) \quad \ell_{m,k} \cdot r_{m',k'} = \begin{cases} 1, & \text{if } m = m' \text{ and } k = k' \\ 0, & \text{otherwise} \end{cases}.$$

For $v \in \mathbb{C}^N$ set

$$(2.49) \quad P_{m,k}v = (\ell_{m,k} \cdot v)r_{m,k} \text{ (no complex conjugation here) .}$$

So now we may write

$$(2.50) \quad \mathbb{E} = \mathbb{E}_0 + \sum_{m=1}^M \sum_{k=1}^{\mu_m} \mathbb{E}_{m,k},$$

where

$$(2.51) \quad \mathbb{E}_{m,k}(V_\alpha e^{i\alpha \cdot \theta}) := \begin{cases} (P_{m,k}V_\alpha)e^{in_\alpha \theta_m}, & \alpha \in \mathcal{C}_m \setminus 0 \\ 0, & \text{otherwise} \end{cases} \quad ; \text{ i.e., } \mathbb{E}_{m,k} = P_{m,k}\mathbb{E}_m.$$

Recall the potential use of the projectors \mathbb{E}_m discussed in Remark 2.10. We will obtain a system of equations for $\underline{v}(x)$, $\sigma_{m,k}(x, \theta_m)$ by applying \mathbb{E}_0 and the $\mathbb{E}_{m,k}$ to (2.41)(b).

We omit proof of the following lemma, which is almost identical to that of Lemma 2.11 of [2].

Lemma 2.14. *Suppose $\mathbb{E}\mathcal{V}^0 = \mathcal{V}^0$. Then*

$$(2.52) \quad \mathbb{E}_{m,k}(\tilde{L}(\partial_x)\mathcal{V}^0) = (X_{\phi_m}\sigma_{m,k})r_{m,k}$$

where X_{ϕ_m} is the characteristic vector field associated to ϕ_m :

$$(2.53) \quad X_{\phi_m} := \partial_{x_d} + \sum_{j=0}^{d-1} -\partial_{\xi_j}\underline{\omega}_m(\beta)\partial_{x_j}.$$

We proceed by defining the interaction integrals which will appear in the interior equations.

Definition 2.15. *Suppose (ϕ_q, ϕ_r) forms a normalized p -resonance, with*

$$(2.54) \quad n_p\phi_p = n_q\phi_q + n_r\phi_r.$$

For any $f \in H_T^s(x, \theta_q)$ define $f_{n_q} \in H_T^s(x, \theta_q)$ to be the image of f under the preparation map

$$(2.55) \quad f(x, \theta_q) = \sum_{k \in \mathbb{Z}} f_k(x) e^{ik\theta_q} \rightarrow \sum_{k \in \mathbb{Z}} f_{kn_q}(x) e^{ikn_q\theta_q}.$$

Suppose $s > \frac{d+3}{2} + 1$ and that $\sigma_{q,k}, \sigma_{r,k'} \in H_T^s(\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T})$. We get exactly two normalized p -resonance formations from the arrangements of (2.54):

$$(2.56) \quad n_p\phi_p = n_q\phi_q + n_r\phi_r, \quad n_p\phi_p = n_r\phi_r + n_q\phi_q.$$

To these equations we associate, respectively, the two families of prepared integrals:

$$\begin{aligned}
J_{p,n_q,n_r}^{k,k'}(x,\theta_p) &:= \\
\frac{1}{2\pi} \int_0^{2\pi} (\sigma_{q,k})_{n_q} \left(x, \frac{n_p}{n_q} \theta_p - \frac{n_r}{n_q} \theta_r \right) \partial_{\theta_r} \sigma_{r,k'}(x,\theta_r) d\theta_r, \quad k \in \{1, \dots, \mu_q\}, k' \in \{1, \dots, \mu_r\}, \\
J_{p,n_r,n_q}^{k,k'}(x,\theta_p) &:= \\
\frac{1}{2\pi} \int_0^{2\pi} (\sigma_{r,k})_{n_r} \left(x, \frac{n_p}{n_r} \theta_p - \frac{n_q}{n_r} \theta_q \right) \partial_{\theta_q} \sigma_{q,k'}(x,\theta_q) d\theta_q, \quad k \in \{1, \dots, \mu_r\}, k' \in \{1, \dots, \mu_q\}.
\end{aligned}
\tag{2.57}$$

Remark 2.16. Strictly speaking, $J_{p,n_q,n_r}^{k,k'}$ and $J_{p,n_r,n_q}^{k,k'}$ are symbolically $I_{n_q,n_p,n_r'}^{k,k'}$ and $I_{n_r',-n_p,n_q}^{k,k'}$ (as defined in [2], with respect to the triple (ϕ_q, ϕ_p, ϕ_r) and $n_q \phi_q = n_p \phi_p + n_r' \phi_r$ where $n_r' = -n_r$). The difference is that we do not require $n_q > 0$, $q < p < r$ – recall, instead, our concern is that $(n_p, n_q, n_r) \in Z_p \times Z_q \times Z_r$.

The following proposition shows that the prepared integrals ‘pick out’ the p -resonances.

Proposition 2.17. Suppose $s > \frac{d+3}{2} + 1$ and that $\sigma_{q,k}, \sigma_{r,k'} \in H_T^s(\mathbb{R}_+^{d+1} \times \mathbb{T})$ have Fourier series

$$\sigma_{q,k}(x, \theta_q) = \sum_{j \in Z_q \setminus 0} a_j(x) e^{ij\theta_q} \text{ and } \sigma_{r,k'}(x, \theta_r) = \sum_{j \in Z_r \setminus 0} b_j(x) e^{ij\theta_r}.
\tag{2.58}$$

Given (ϕ_q, ϕ_r) forms a normalized p -resonance, the prepared integral $J_{p,n_q,n_r}^{k,k'}(x, \theta_p)$ belongs to $H_T^{s-1}(x, \theta_p)$ and has Fourier series

$$J_{p,n_q,n_r}^{k,k'}(x, \theta_p) = \sum_{j \in \mathbb{Z}} a_{jn_q}(x) b_{jn_r}(x) i \cdot (jn_r) e^{ijn_p \theta_p}.
\tag{2.59}$$

For the other integral in (2.57), one switches n_q and n_r above. Moreover, the functions $J_{p,n_q,n_r}^{k,k'}(x, \theta_p)$ can be described for $\theta_p \in \mathbf{C}_p$ through analytic extension of the expansions into the complex half plane \mathbf{C}_p in the same manner that the profiles $\sigma_{m,k}(x, \theta_m)$ are extended into \mathbf{C}_m in Section 1.2.1.

Remark 2.18. We note that for $p \in \mathcal{P} \cup \mathcal{N}$, in (2.59), we can sum over just $j \in \mathbb{Z}^+ \setminus 0$. To see this, observe that then (ϕ_q, ϕ_r) forms an elliptic resonance, so Proposition 2.13 guarantees one of ϕ_q, ϕ_r is elliptic – without loss of generality, say ϕ_q . Thus, $Z_q \setminus 0$ consists only of negative integers or positive integers, and the only nonzero Fourier coefficients of $\sigma_{q,k}$ are those $a_j(x)$ appearing in (2.58), with $j \in Z_q \setminus 0$; we have $a_j(x) = 0$ for all other j . For example, if $q \in \mathcal{P}$, then $Z_q = \mathbb{Z}^+$, and so, noting $n_q > 0$, we have the $a_{jn_q}(x)$ appearing in (2.59) are only nonzero for $j > 0$. Indeed, $q \in \mathcal{N}$ implies the same of the $a_{jn_q}(x)$, so we may sum over just $j \in \mathbb{Z}^+ \setminus 0$ in either case. We add that, for such j , the $e^{ijn_p \theta_p}$ factors are bounded as θ_p ranges over \mathbf{C}_p .

Proof of Proposition 2.17. The main part of the proof is showing that the expansion (2.59) converges to the desired result in $H_T^{s-1}(x, \theta_p)$, where we have restricted our attention to real θ_p . The proof of this is very similar to the proof of Proposition 2.13 from [2]. Analytic extension into the complex half plane \mathbf{C}_p is done in the same way that the expansions for the profiles are extended into their corresponding complex half planes, as discussed in Section 1.2.1. \square

Interior equations.

By applying \mathbb{E}_0 and the $\mathbb{E}_{m,k}$ to (2.41)(b), we obtain the system for $\underline{v}(x)$ and the $\sigma_{m,k}(x, \theta_m)$ with the following proposition.

Proposition 2.19. Suppose $(\phi_{\underline{q}}, \phi_{\underline{r}})$ forms a normalized \underline{p} -resonance, with

$$n_{\underline{p}} \phi_{\underline{p}} = n_{\underline{q}} \phi_{\underline{q}} + n_{\underline{r}} \phi_{\underline{r}}.
\tag{2.60}$$

and that all other normalized triples of resonant phases are permutations of $(\phi_{\underline{p}}, \phi_{\underline{q}}, \phi_{\underline{r}})$ each corresponding to one of the six rearrangements of (2.60).

(i) Given \mathcal{V}^0 as in (2.36) and represented by (2.9), the equation (2.41)(b) is equivalent to the following system:

$$(2.61) \quad \tilde{L}(\partial_x)\underline{v} + \sum_{j=0}^{d-1} \sum_{m=1}^M \sum_{k,k'=1}^{\mu_m} \frac{1}{2\pi} \left(\int_0^{2\pi} \sigma_{m,k}(x, \theta_m) \partial_{\theta_m} \sigma_{m,k'}(x, \theta_m) d\theta_m \right) R_{j,m}^{k,k'} = F(0)\underline{v};$$

(2.62)

$$\begin{aligned} (a) & X_{\phi_p} \sigma_{p,l}(x, \theta_p) + \sum_{j=0}^{d-1} \sum_{k'=1}^{\mu_p} a_{p,l,j}^{k'}(\underline{v}) \partial_{\theta_p} \sigma_{p,k'}(x, \theta_p) + \\ (b) & \sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} b_{p,l,j}^{k,k'} \sigma_{p,k}(x, \theta_p) \partial_{\theta_p} \sigma_{p,k'}(x, \theta_p) - \sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} b_{p,l,j}^{k,k'} \frac{1}{2\pi} \int_0^{2\pi} \sigma_{p,k}(x, \theta_p) \partial_{\theta_p} \sigma_{p,k'}(x, \theta_p) d\theta_p + \\ (c) & \sum_{j=0}^{d-1} J_{p,l,j}(x, \theta_p) = \sum_{k=1}^{\mu_p} e_{p,l}^k \sigma_{p,k}(x, \theta_p) \end{aligned}$$

for $p \in \mathcal{I} \cup \mathcal{O}$, $l \in \{1, \dots, \mu_p\}$,

where

$$(2.63) \quad J_{p,l,j}(x, \theta_p) = \begin{cases} \sum_{k=1}^{\mu_{\underline{q}}} \sum_{k'=1}^{\mu_{\underline{r}}} c_{p,l,j}^{k,k'} J_{\underline{p},n_{\underline{q}},n_{\underline{r}}}^{k,k'}(x, \theta_p) + \sum_{k=1}^{\mu_{\underline{r}}} \sum_{k'=1}^{\mu_{\underline{q}}} d_{p,l,j}^{k,k'} J_{\underline{p},n_{\underline{r}},n_{\underline{q}}}^{k,k'}(x, \theta_p), & p = \underline{p}, \\ \sum_{k=1}^{\mu_{\underline{p}}} \sum_{k'=1}^{\mu_{\underline{r}}} c_{p,l,j}^{k,k'} J_{\underline{q},-n_{\underline{p}},n_{\underline{r}}}^{k,k'}(x, \theta_p) + \sum_{k=1}^{\mu_{\underline{r}}} \sum_{k'=1}^{\mu_{\underline{p}}} d_{p,l,j}^{k,k'} J_{\underline{q},n_{\underline{r}},-n_{\underline{p}}}^{k,k'}(x, \theta_p), & p = \underline{q}, \\ \sum_{k=1}^{\mu_{\underline{q}}} \sum_{k'=1}^{\mu_{\underline{p}}} c_{p,l,j}^{k,k'} J_{\underline{r},n_{\underline{q}},-n_{\underline{p}}}^{k,k'}(x, \theta_p) + \sum_{k=1}^{\mu_{\underline{p}}} \sum_{k'=1}^{\mu_{\underline{q}}} d_{p,l,j}^{k,k'} J_{\underline{r},-n_{\underline{p}},n_{\underline{q}}}^{k,k'}(x, \theta_p), & p = \underline{r}, \\ 0, & \text{otherwise;} \end{cases}$$

$$\begin{aligned} (a) & X_{\phi_p} \sigma_{p,l}(x, \theta_p) + \sum_{j=0}^{d-1} \sum_{k'=1}^{\mu_p} a_{p,l,j}^{k'}(\underline{v}) \partial_{\theta_p} \sigma_{p,k'}(x, \theta_p) + \\ (b) & \sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} b_{p,l,j}^{k,k'} \sigma_{p,k}(x, \theta_p) \partial_{\theta_p} \sigma_{p,k'}(x, \theta_p) + \\ (c) & \sum_{j=0}^{d-1} J_{p,l,j}(x, \theta_p) = \sum_{k=1}^{\mu_p} e_{p,l}^k \sigma_{p,k}(x, \theta_p) \end{aligned}$$

for $p \in \mathcal{P} \cup \mathcal{N}$, $l \in \{1, \dots, \mu_p\}$,

where

$$(2.65) \quad J_{p,l,j}(x, \theta_p) = \begin{cases} \sum_{k=1}^{\mu_{\underline{q}}} \sum_{k'=1}^{\mu_{\underline{r}}} c_{p,l,j}^{k,k'} J_{\underline{p},n_{\underline{q}},n_{\underline{r}}}^{k,k'}(x, \theta_p) + \sum_{k=1}^{\mu_{\underline{r}}} \sum_{k'=1}^{\mu_{\underline{q}}} d_{p,l,j}^{k,k'} J_{\underline{p},n_{\underline{r}},n_{\underline{q}}}^{k,k'}(x, \theta_p), & p = \underline{p}, \\ 0, & \text{otherwise.} \end{cases}$$

The constant vectors $R_{j,m}^{k,k'}$ appearing in (2.61) are given by

$$(2.66) \quad R_{j,m}^{k,k'} = \beta_j (\partial_u \tilde{A}_j(0) \cdot r_{m,k}) r_{m,k'}.$$

The constant scalars $b_{p,l,j}^{k,k'}$, $c_{p,l,j}^{k,k'}$, $d_{p,l,j}^{k,k'}$, $e_{p,l}^k$, and the coefficients of the scalar linear function of \underline{v} , $a_{p,l,j}^{k'}(\underline{v})$, in (2.62)-(2.65) are given by similar formulas, but now involving dot products with the vector $\ell_{p,l}$.

(ii) Equations (2.61), (2.62), form a hyperbolic subsystem, that is, a system in \underline{v} and $\sigma_{p,l}$ for $p \in \mathcal{I} \cup \mathcal{O}$ independent of $\sigma_{q,k}$ for all $q \in \mathcal{P} \cup \mathcal{N}$.

Remark 2.20. While the hyperbolic interior equations are independent of the elliptic profiles, the elliptic interior equations (2.64) are not in general independent of the hyperbolic profiles, since these appear in the elliptic interaction integrals if some phase paired with a hyperbolic phase forms an elliptic resonance.

Proof of Proposition 2.19. Regarding (i), (2.61) follows from application of \mathbb{E}_0 to (2.41)(b), and (2.62) from application of $\mathbb{E}_{p,l}$ to (2.41)(b). One similarly arrives at (2.64): in doing so, one finds that the second triple sum appearing in (2.62)(b) is zero for $p \in \mathcal{P} \cup \mathcal{N}$, since then $\text{spec } \sigma_{p,k}, \text{spec } \sigma_{p,k'} \subset Z_p \setminus 0$ which is either $\mathbb{Z}^+ \setminus 0$ or $\mathbb{Z}^- \setminus 0$, resulting in (2.64)(b). The differences between (2.63) and (2.65) merely account for the fact that for $p \in \mathcal{I} \cup \mathcal{O}$, we have $n_p \in Z_p \setminus 0 = \mathbb{Z} \setminus 0$ implies $-n_p \in Z_p \setminus 0 = \mathbb{Z} \setminus 0$, while for $p \in \mathcal{P} \cup \mathcal{N}$, we have $n_p \in Z_p \setminus 0$ implies $-n_p \notin Z_p \setminus 0$. The reverse implication of (i) follows upon recalling the decomposition (2.50).

For proof of (ii), first we show the elliptic profiles drop out of (2.61). This follows from the argument made in proving (i) that terms such as those in the triple sum of (2.61) with $m \in \mathcal{P} \cup \mathcal{N}$ are zero, since then $\text{spec } \sigma_{m,k}, \text{spec } \sigma_{m,k'} \subset Z_p \setminus 0$. Finally, to see that no $\sigma_{q,k}$ with $q \in \mathcal{P} \cup \mathcal{N}$ appears in (2.62), we have only to check the interaction integral terms $J_{p,l,j}$ of (2.62)(c). However, the terms of (2.63) defining $J_{p,l,j}$ involve only profiles corresponding to p -resonant phases; $J_{p,l,j}$ is only nonzero for some $p \in \mathcal{I} \cup \mathcal{O}$ if $p = \underline{p}, \underline{q},$ or \underline{r} and, by Proposition 2.13, $\underline{p}, \underline{q}, \underline{r} \in \mathcal{I} \cup \mathcal{O}$. Thus, in this case, no elliptic profiles appear in $J_{p,l,j}$. \square

From the above, we find the hyperbolic subsystem has the same form as the system in [2] and we will solve it similarly. On the other hand, we will not solve the elliptic interior equations exactly.

Boundary equations.

Now that we have the interior equations for the profiles, we formulate the boundary conditions to be imposed on the profiles. From (2.41)(c), we get

$$(2.67) \quad \begin{aligned} (a) \quad & B(0)\underline{v} = \underline{G}(x') \\ (b) \quad & B(0)\mathcal{V}^{0*}(x', 0, \theta_0, \dots, \theta_0) = \\ & B(0) \left(\sum_{m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}} \sum_{k=1}^{\nu_{km}} \sigma_{m,k}(x', 0, \theta_0) r_{m,k} + \sum_{m \in \mathcal{O}} \sum_{k=1}^{\nu_{km}} \sigma_{m,k}(x', 0, \theta_0) r_{m,k} \right) = G^*(x', \theta_0), \end{aligned}$$

where \mathcal{V}^{0*} denotes the mean zero part of the periodic function \mathcal{V}^0 , and we have similar for G^* . While (2.67)(a) gives boundary data for \underline{v} , it remains to establish satisfactory boundary conditions on the individual profiles $\sigma_{m,k}$ from (2.67)(b). This is done with Proposition 2.22.

Lemma 2.21. *Each of the sets of vectors*

$$(2.68) \quad \{B(0)r_{m,k} : m \in \mathcal{I} \cup \mathcal{P}, k = 1, \dots, \mu_m\},$$

$$(2.69) \quad \{B(0)r_{m,k} : m \in \mathcal{I} \cup \mathcal{N}, k = 1, \dots, \mu_m\},$$

is a basis of $\mathbb{C}^{\mathcal{P}}$.

Proof. First we prove (2.68) is a basis of $\mathbb{C}^{\mathcal{P}}$. Recall our assumption of uniform stability, Assumption (1.5), which tells us that $B(0)$ maps a basis for the stable subspace $\mathbb{E}^s(\underline{\mathcal{I}}, \underline{\eta})$ to a basis for $\mathbb{C}^{\mathcal{P}}$. Thus, according to Lemma 1.13, which states that

$$(2.70) \quad \mathbb{E}^s(\underline{\mathcal{I}}, \underline{\eta}) = \oplus_{m \in \mathcal{I} \cup \mathcal{P}} \text{Ker } L(d\phi_m),$$

we have that the set (2.68) is a basis for $\mathbb{C}^{\mathcal{P}}$. To see the same holds for (2.69), recall from Remark 1.10(ii) the fact that for each nonreal $\underline{\omega}_m$, say, without loss of generality, $m \in \mathcal{P}$, there is another eigenvalue $\underline{\omega}_{m'}$ satisfying $\underline{\omega}_{m'} = \overline{\underline{\omega}_m}$, thus with $m' \in \mathcal{N}$, and that similar holds for the associated eigenvectors. It follows that

$$(2.71) \quad \{B(0)r_{m,k} : m \in \mathcal{I} \cup \mathcal{N}, k = 1, \dots, \mu_m\} = \{B(0)\bar{r}_{m,k} : m \in \mathcal{I} \cup \mathcal{P}, k = 1, \dots, \mu_m\}$$

$$(2.72) \quad = \{\overline{B(0)r_{m,k}} : m \in \mathcal{I} \cup \mathcal{P}, k = 1, \dots, \mu_m\}.$$

Clearly, (2.72) spans the same subspace as (2.68), i.e. $\mathbb{C}^{\mathcal{P}}$, and so (2.69) is also a basis of $\mathbb{C}^{\mathcal{P}}$. \square

Proposition 2.22. *The data $(\sigma_{m,k}(x', 0, \theta_0); m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}, k \in \{1, \dots, \mu_m\})$ are determined by the data $(\sigma_{m,k}(x', 0, \theta_0); m \in \mathcal{O}, k \in \{1, \dots, \mu_m\})$ in that there exist constant matrices \mathbb{M}^\pm such that the zero-mean boundary condition,*

$$(2.73) \quad B(0)\mathcal{V}^{0*}(x', 0, \theta_0, \dots, \theta_0) = B(0) \left(\sum_{m=1}^M \sum_{k=1}^{\mu_m} \sigma_{m,k,n}(x', 0, \theta_0) r_{m,k} \right) = G^*(x', \theta_0),$$

is equivalent to the condition

$$(2.74) \quad (\sigma_{m,k}^\pm(x', 0, \theta_0); m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}, k \in \{1, \dots, \mu_m\}) = \mathbb{M}^\pm(G^{*,\pm}, \sigma_{m,k}^\pm(x', 0, \theta_0); m \in \mathcal{O}, k \in \{1, \dots, \mu_m\}),$$

(using $+$ ($-$) to denote a part with positive (negative) spectrum.)

Proof. We rewrite (2.73) as

$$(2.75) \quad B(0)\mathcal{V}^{0*}(x', 0, \theta_0, \dots, \theta_0) = B(0) \left(\sum_{m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}} \sum_{k=1}^{\mu_m} \sigma_{m,k}(x', 0, \theta_0) r_{m,k} + \sum_{m \in \mathcal{O}} \sum_{k=1}^{\mu_m} \sigma_{m,k}(x', 0, \theta_0) r_{m,k} \right) = G^*(x', \theta_0).$$

Recall \mathcal{P} -profiles have positive spectra and \mathcal{N} -profiles have negative spectra. Thus, using the subscript n to denote the n th term in a Fourier series, we get for $n > 0$,

$$(2.76) \quad B(0) \left(\sum_{m \in \mathcal{I} \cup \mathcal{P}} \sum_{k=1}^{\mu_m} \sigma_{m,k,n}(x', 0, \theta_0) r_{m,k} \right) = G_n^*(x', \theta_0) - B(0) \left(\sum_{m \in \mathcal{O}} \sum_{k=1}^{\mu_m} \sigma_{m,k,n}(x', 0, \theta_0) r_{m,k} \right),$$

and for $n < 0$,

$$(2.77) \quad B(0) \left(\sum_{m \in \mathcal{I} \cup \mathcal{N}} \sum_{k=1}^{\mu_m} \sigma_{m,k,n}(x', 0, \theta_0) r_{m,k} \right) = G_n^*(x', \theta_0) - B(0) \left(\sum_{m \in \mathcal{O}} \sum_{k=1}^{\mu_m} \sigma_{m,k,n}(x', 0, \theta_0) r_{m,k} \right).$$

By Lemma 2.21, both sets $\{B(0)r_{m,k} : k \in \{1, \dots, \mu_m\}, m \in \mathcal{I} \cup \mathcal{P}\}$, $\{B(0)r_{m,k} : k \in \{1, \dots, \mu_m\}, m \in \mathcal{I} \cup \mathcal{N}\}$ are bases for \mathbb{C}^p , and the desired result follows from this. \square

So we now have a sub-system ((2.61), (2.62)) in just the hyperbolic profiles $\sigma_{m,k}$, $m \in \mathcal{I} \cup \mathcal{O}$, and the mean \underline{v} with boundary data for \underline{v} and determination of \mathcal{I} -boundary data from \mathcal{O} -boundary data as in (2.67)(b), where (2.67)(b) is equivalent to (2.74).

Remark 2.23. We will seek \underline{v} , $\sigma_{p,l}$ such that, in addition to the equations above, we satisfy the initial conditions

$$(2.78) \quad \underline{v} = 0 \quad \text{and} \quad \sigma_{p,l} = 0 \quad \text{in } t \leq 0 \quad \text{for all } p, l.$$

The large system consists of the interior equations (2.61)-(2.65) with boundary equations (2.67) and the initial conditions (2.78).

2.3 Solution of the hyperbolic sub-system of the large system.

First we will obtain the ‘hyperbolic part’ of our solution, \mathcal{V}_h^0 , which will satisfy, in place of (2.41),

$$(2.79) \quad \begin{aligned} a) & \mathbb{E}_h \mathcal{V}_h^0 = \mathcal{V}_h^0 \\ b) & \mathbb{E}_h \left(\tilde{L}(\partial_x) \mathcal{V}_h^0 + \mathcal{M}(\mathcal{V}_h^0) \partial_\theta \mathcal{V}_h^0 \right) = \mathbb{E}_h(F(0) \mathcal{V}_h^0) \quad \text{in } x_d \geq 0 \\ c) & B(0) \mathcal{V}_h^0(x', 0, \theta_0, \dots, \theta_0) = G(x', \theta_0) \\ d) & \mathcal{V}_h^0 = 0 \quad \text{in } t < 0. \end{aligned}$$

Condition (2.79)(a) serves the same purpose as (2.41)(a), but adds the restriction that \mathcal{V}_h^0 only consists of the mean and the hyperbolic profiles. Thus, recalling the form for \mathcal{V}^0 , we see \mathcal{V}_h^0 takes the form

$$(2.80) \quad \mathcal{V}_h^0(x, \theta_1, \dots, \theta_M) = \underline{v}(x) + \sum_{m \in \mathcal{I} \cup \mathcal{O}} \sum_{k=1}^{\mu_m} \sigma_{m,k}(x, \theta_m) r_{m,k}.$$

The condition (2.79)(b) is exactly the hyperbolic sub-system ((2.61),(2.62)) found in Proposition 2.19. This follows directly from Proposition 2.19 and the definition of \mathbb{E}_h . Since \mathcal{V}_h^0 has the form (2.80), the boundary condition (2.79)(c) becomes (2.74), ignoring the appearance of the elliptic profiles ($\sigma_{m,k}$ with $m \in \mathcal{P} \cup \mathcal{N}$) in the left hand side.

To solve this, we employ the same approach as that which is used in [2] to solve the full system of profile equations. This is done by solving the system with an iteration scheme such as the following

$$(2.81) \quad \begin{aligned} & a) \mathbb{E}_h \mathcal{V}_h^{0,n} = \mathcal{V}_h^{0,n} \\ & b) \mathbb{E}_h \left(\tilde{L}(\partial_x) \mathcal{V}_h^{0,n} + \mathcal{M}(\mathcal{V}_h^{0,n-1}) \partial_\theta \mathcal{V}_h^{0,n} \right) = \mathbb{E}_h(F(0) \mathcal{V}_h^{0,n-1}) \text{ in } x_d \geq 0 \\ & c) B(0) \mathcal{V}_h^{0,n}(x', 0, \theta_0, \dots, \theta_0) = G(x', \theta_0) \\ & d) \mathcal{V}_h^{0,n} = 0 \text{ in } t < 0, \end{aligned}$$

where we note that (2.81)(a) means $\mathcal{V}_h^{0,n}$ is of the form

$$(2.82) \quad \mathcal{V}_h^{0,n}(x, \theta_1, \dots, \theta_M) = \underline{v}^n(x) + \sum_{m \in \mathcal{I} \cup \mathcal{O}} \sum_{k=1}^{\mu_m} \sigma_{m,k}^n(x, \theta_m) r_{m,k},$$

where iterates $\underline{v}^n(x)$, $\sigma_{m,k}^n(x, \theta_m)$ for $m \in \mathcal{I} \cup \mathcal{O}$ satisfy an iterated version of the hyperbolic sub-system of the profile equations, ((2.61),(2.62)) which is encapsulated by (2.81)(b).

The following proposition gives the solution to the iterated system as well as the solution to the hyperbolic sub-system itself. We omit the proofs of these as they are almost identical to the proof of Proposition 2.19 together with that of Proposition 2.21 from [2].

Proposition 2.24. *Let $T > 0$, $m > \frac{d+3}{2} + 1$ and suppose that $G(x', \theta_0) \in H_T^m$.*

- (i) *Setting $\mathcal{V}_h^{0,0} = 0$, there exist unique iterates $\mathcal{V}_h^{0,n} \in \mathbb{H}_T^m$, $n \geq 1$, solving the system (2.81).*
- (ii) *For some $0 < T_0 \leq T$ the system (2.79) has a unique solution $\mathcal{V}_h^0 \in \mathbb{H}_{T_0}^m$. Furthermore,*

$$(2.83) \quad \lim_{n \rightarrow \infty} \mathcal{V}_h^{0,n} = \mathcal{V}_h^0 \text{ in } \mathbb{H}_{T_0}^{m-1},$$

and the traces $\mathcal{V}_h^0|_{x_d=0}$, $\underline{v}|_{x_d=0}$, and $\sigma_{p,l}|_{x_d=0}$, $p \in \mathcal{I} \cup \mathcal{O}$, all lie in $H_{T_0}^m$.

2.4 Approximate solution of the equations for the elliptic profiles.

Recall from Proposition 2.19 the elliptic interior equations:

$$(2.84) \quad \begin{aligned} & (a) X_{\phi_p} \sigma_{p,l}(x, \theta_p) + \sum_{j=0}^{d-1} \sum_{k'=1}^{\mu_p} a_{p,l,j}^{k'}(\underline{v}) \partial_{\theta_p} \sigma_{p,k'}(x, \theta_p) + \\ & (b) \sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} b_{p,l,j}^{k,k'} \sigma_{p,k}(x, \theta_p) \partial_{\theta_p} \sigma_{p,k'}(x, \theta_p) + \\ & (c) \sum_{j=0}^{d-1} J_{p,l,j}(x, \theta_p) = \sum_{k=1}^{\mu_p} e_{p,l}^k \sigma_{p,k}(x, \theta_p) \\ & \text{for } p \in \mathcal{P} \cup \mathcal{N}, l \in \{1, \dots, \mu_p\}. \end{aligned}$$

These complex transport equations may not generally have exact solutions. Instead, our elliptic profiles $\sigma_{p,l}$ will approximately solve these equations in the sense that they will hold at the boundary $x_d = 0$. It is

sufficient to simply evaluate the above expression at $x_d = 0$, isolate $\partial_{x_d}\sigma_{p,l}(x', 0, \theta_p)$, and require that $\sigma_{p,l}$ has the appropriate x_d -derivative at the boundary. Of course, we will also have to adhere to the boundary conditions, so that the trace $\sigma_{p,l}(x', 0, \theta_0)$ at the boundary satisfies (2.67)(b), and we must satisfy initial conditions (2.78). With Proposition 2.30, we construct such elliptic profiles $\sigma_{p,l}(x, \theta_p)$ by solving some wave equations in which x_d plays the role of the time variable and with appropriate boundary data corresponding to $\sigma_{p,l}(x', 0, \theta_p)$ and $\partial_{x_d}\sigma_{p,l}(x', 0, \theta_p)$.

In addition to the exact elliptic interior equations, we consider semilinear equations in iterates $\sigma_{p,l}^n$ for $p \in \mathcal{P} \cup \mathcal{N}$, $l \in \{1, \dots, \mu_p\}$, almost identical to the iteration scheme which is used to obtain the hyperbolic profiles. While we do not need elliptic iterates $\sigma_{p,l}^n$ to get the elliptic profiles $\sigma_{p,l}$, which are instead obtained by the process described above, they allow us to handle hyperbolic parts and elliptic parts uniformly in the error analysis. The semilinear equations in the elliptic iterates $\sigma_{p,l}^n$ are

$$(2.85) \quad X_{\phi_p}\sigma_{p,l}^n(x, \theta_p) +$$

$$(2.86) \quad \sum_{j=0}^{d-1} \sum_{k'=1}^{\mu_p} a_{p,l,j}^{k'}(\psi^{n-1}) \partial_{\theta_p} \sigma_{p,k'}^n(x, \theta_p) + \sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} b_{p,l,j}^{k,k'} \sigma_{p,k}^{n-1}(x, \theta_p) \partial_{\theta_p} \sigma_{p,k'}^n(x, \theta_p) +$$

$$(2.87) \quad \sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} c_{p,l,j}^{k,k'} J_{p,n_q,n_r}^{k,k',n}(x, \theta_p) + \sum_{j=0}^{d-1} \sum_{k=1}^{\mu_p} \sum_{k'=1}^{\mu_p} d_{p,l,j}^{k,k'} J_{p,n_r,n_q}^{k,k',n}(x, \theta_p)$$

$$(2.88) \quad = \sum_{k=1}^{\mu_p} e_{p,l}^k \sigma_{p,k}^{n-1}(x, \theta_p), \quad \text{for } p \in \mathcal{P} \cup \mathcal{N}, \quad l \in \{1, \dots, \mu_p\},$$

where we define

$$(2.89) \quad J_{p,n_q,n_r}^{k,k',n}(x, \theta_p) = \frac{1}{2\pi} \int_0^{2\pi} (\sigma_{q,k}^{n-1})_{n_q} \left(x, \frac{n_p}{n_q} \theta_p - \frac{n_r}{n_q} \theta_r \right) \partial_{\theta_r} \sigma_{r,k'}^n(x, \theta_r) d\theta_r,$$

and similar for $J_{p,n_r,n_q}^{k,k',n}$. Imposed on the iterates $\sigma_{p,l}^n$ are boundary conditions identical to those for the profiles $\sigma_{p,l}$, i.e. (2.74) with $\sigma_{m,k}^n$ in place of each $\sigma_{m,k}$, and initial conditions

$$(2.90) \quad \sigma_{p,l}^n(x) = 0 \text{ for } t \leq 0.$$

Similar to the elliptic profiles, the elliptic iterates $\sigma_{p,l}^n$ will only approximately solve (2.85)-(2.88) in the sense that these equations will hold at $x_d = 0$. In fact, the elliptic iterates $\sigma_{p,l}^n$ are obtained with the same kind of construction used for the elliptic profiles. In addition to the elliptic profiles, their corresponding iterates are constructed in Proposition 2.30.

For now, let us fix $p \in \{1, \dots, M\}$, $l \in \{1, \dots, \mu_p\}$ and an integer n and take the task of finding approximate solutions $\sigma_{p,l}$ and $\sigma_{p,l}^n$ of (2.84) and the system (2.85)-(2.88), respectively. First we rearrange (2.84), isolating the vector field applied to $\sigma_{p,l}$ in the left hand side, and denoting the resulting right hand side by $f_{p,l}$, getting something of the form

$$(2.91) \quad X_{\phi_p}\sigma_{p,l}(x, \theta_p) = f_{p,l}(x, \theta_p).$$

Similarly, we isolate $X_{\phi_p}\sigma_{p,l}^n$ in the left hand side of (2.85)-(2.88), and define $f_{p,l}^n(x, \theta_p)$ to be what remains on the right hand side, so (2.84) becomes

$$(2.92) \quad X_{\phi_p}\sigma_{p,l}^n(x, \theta_p) = f_{p,l}^n(x, \theta_p).$$

Now we isolate the quantities this prescribes for the traces of the $\sigma_{p,l}$ and the $\sigma_{p,l}^n$ at the boundary $x_d = 0$.

Definition 2.25. (i) The coefficient of $\partial_{x_d}\sigma_{p,l}$ in the left hand side of (2.91) is one, so we may isolate it in this expression. We will require this equation to hold at $x_d = 0$, and define $b_{p,l}$ so that doing so is represented by the condition

$$(2.93) \quad \partial_{x_d}\sigma_{p,l}|_{x_d=0} = b_{p,l},$$

noting that the right hand side depends only on the boundary data $\sigma_{m,k}|_{x_d=0}$, $m = 1, \dots, M$, $k = 1, \dots, \mu_m$.

(ii) We define $a_{p,l}$ such that the boundary condition on $\sigma_{p,l}$ in (2.74) is equivalent to

$$(2.94) \quad \sigma_{p,l}|_{x_d=0} = a_{p,l},$$

noting that the right hand side has already been determined by our solution of the hyperbolic profiles. This thus determines the right hand side of (2.93).

(iii) We isolate $\partial_{x_d}\sigma_{p,l}^n$ on the left hand side of (2.85)-(2.88), and define $b_{p,l}^n$ such that requiring this to hold at $x_d = 0$ is equivalent to

$$(2.95) \quad \partial_{x_d}\sigma_{p,l}^n|_{x_d=0} = b_{p,l}^n.$$

(iv) Consider

$$(2.96) \quad (\sigma_{m,k}^{n,\pm}(x', 0, \theta_0); m \in \mathcal{I} \cup \mathcal{P} \cup \mathcal{N}, k \in \{1, \dots, \mu_m\}) = \mathbb{M}^\pm(G^{*,\pm}, \sigma_{m,k}^{n,\pm}(x', 0, \theta_0); m \in \mathcal{O}, k \in \{1, \dots, \mu_m\}),$$

for \mathbb{M}^\pm as defined in the proof of Proposition 2.22. We define $a_{p,l}^n$ such that the condition on $\sigma_{p,l}^n$ in (2.96) is equivalent to

$$(2.97) \quad \sigma_{p,l}^n|_{x_d=0} = a_{p,l}^n.$$

Lemma 2.26. Suppose $G \in H_{T_0}^{s+1}$. Let $\mathcal{V}_h^0 \in \mathbb{H}_{T_0}^{s+1}$ be the solution of the hyperbolic sub-system constructed in Proposition 2.24. For $a_{p,l}^n$, $b_{p,l}^n$, $a_{p,l}$, and $b_{p,l}$ as defined in Definition 2.25, we have $a_{p,l}^n \in H_{T_0}^{s+1}(x', \theta_p)$, $b_{p,l}^n \in H_{T_0}^s(x', \theta_p)$, and

$$(2.98) \quad \lim_{n \rightarrow \infty} a_{p,l}^n = a_{p,l} \text{ in } H_{T_0}^{s+1}(x', \theta_p),$$

$$(2.99) \quad \lim_{n \rightarrow \infty} b_{p,l}^n = b_{p,l} \text{ in } H_{T_0}^s(x', \theta_p).$$

Proof. Checking $a_{p,l}^n \in H_{T_0}^{s+1}(x', \theta_p)$ is straightforward, since $G(x', \theta_p)$, $\sigma_{m,k}^n(x', 0, \theta_p)$, $m \in \mathcal{O}$, are in $H_{T_0}^{s+1}(x', \theta_p)$.

Now we show $b_{p,l}^n \in H_{T_0}^s(x', \theta_p)$. Observe $b_{p,l}^n$ consists of the terms evaluated at $x_d = 0$ (2.86), (2.87), (2.88), and

$$(2.100) \quad (\partial_{x_d} - X_{\phi_p})\sigma_{p,l}^n(x', 0, \theta_p).$$

Since the $\underline{v}_{p,k}^n$, $\sigma_{p,k}^n(x', 0, \theta_p)$, $p \in \mathcal{I} \cup \mathcal{O}$, are in $H_{T_0}^{s+1}(x', \theta_p)$, and $H_{T_0}^s(x', \theta_p)$ is a Banach algebra, it follows that (2.100) and both (2.86) and (2.88) at $x_d = 0$ are in $H_{T_0}^s(x', \theta_p)$. It remains to show this for (2.87) at $x_d = 0$. This follows from Proposition 2.17 with x' in place of x . Taking into account that, for $p \in \mathcal{I} \cup \mathcal{O}$,

$$(2.101) \quad \lim_{n \rightarrow \infty} \sigma_{p,k}^n|_{x_d=0} = \sigma_{p,k}|_{x_d=0} \text{ in } H_{T_0}^{s+1}(x', \theta_p),$$

$$(2.102) \quad \lim_{n \rightarrow \infty} \partial_{x_d}\sigma_{p,k}^n|_{x_d=0} = \partial_{x_d}\sigma_{p,k}|_{x_d=0} \text{ in } H_{T_0}^s(x', \theta_p),$$

the proof that (2.98) and (2.99) hold is similar. \square

Remark 2.27. To simplify obtaining our approximate solutions of (2.85)-(2.88), it will be useful to extend functions $a_{p,l}^n, a_{p,l} \in H_{T_0}^{s+1} = H^{s+1}((-\infty, T_0) \times \mathbb{R}^{d-1} \times \mathbb{T})$, $b_{p,l}^n, b_{p,l} \in H_{T_0}^s = H^s((-\infty, T_0) \times \mathbb{R}^{d-1} \times \mathbb{T})$ on the half-space to elements of $H^{s+1}(\mathbb{R}^d \times \mathbb{T})$ and $H^s(\mathbb{R}^d \times \mathbb{T})$, respectively.

Lemma 2.28. For $s \geq 0$ there is a continuous extension map

$$(2.103) \quad E : H^s((-\infty, T_0) \times \mathbb{R}^{d-1} \times \mathbb{T}) \rightarrow H^s(\mathbb{R}^d \times \mathbb{T}).$$

Proof. It is shown in 4.4 of [9] that there is a continuous extension map from $H^s(\mathbb{R}_+^d)$ to $H^s(\mathbb{R}^d)$. \square

From now on, in place of $a_{p,l}^n, a_{p,l}, b_{p,l}^n, b_{p,l}$ we refer to their extensions to the respective spaces mentioned above unless we explicitly state otherwise.

The following lemma is a standard kind of result in the theory of hyperbolic initial/boundary value problems.

Lemma 2.29. (i) For $a \in H^{s+1}(\mathbb{R}^d \times \mathbb{T})$ and $b \in H^s(\mathbb{R}^d \times \mathbb{T})$, there exists unique $\varsigma \in H_D^{s+1} = H^{s+1}(\mathbb{R}^d \times [0, D] \times \mathbb{T})$ solving the wave equation

$$(2.104) \quad \partial_{x_d}^2 \varsigma - \Delta_{x', \theta_0} \varsigma = 0,$$

and initial- x_d conditions

$$(2.105) \quad \varsigma|_{x_d=0} = a,$$

$$(2.106) \quad \partial_{x_d} \varsigma|_{x_d=0} = b.$$

(ii) Furthermore, ς satisfies

$$(2.107) \quad |\varsigma|_{H_D^{s+1}} \leq C(|a|_{H^{s+1}} + |b|_{H^s}),$$

for some constant C independent of the choice of initial data $\{a, b\}$.

Justification of Lemma 2.29 can be found in Chapter 6 (see 6.18 and Remark 6.21) of [3].

Proposition 2.30. For $a_{p,l}^n, b_{p,l}^n, a_{p,l}$, and $b_{p,l}$ as defined in Definition 2.25, where $p \in \mathcal{P} \cup \mathcal{N}$ there exist $\sigma_{p,l}^n(x, \theta_p), \sigma_{p,l}(x, \theta_p) \in H_{T_0}^{s+1}(x, \theta_p)$ with compact x_d -support in $[0, D]$ satisfying

$$(2.108) \quad \sigma_{p,l}^n = \sigma_{p,l} = 0, \text{ for } t \leq 0,$$

boundary conditions

$$(2.109) \quad \begin{aligned} \sigma_{p,l}^n|_{x_d=0} &= a_{p,l}^n, & \sigma_{p,l}|_{x_d=0} &= a_{p,l}, \\ \partial_{x_d} \sigma_{p,l}^n|_{x_d=0} &= b_{p,l}^n, & \partial_{x_d} \sigma_{p,l}|_{x_d=0} &= b_{p,l}, \end{aligned}$$

and

$$(2.110) \quad \lim_{n \rightarrow \infty} \sigma_{p,l}^n = \sigma_{p,l} \text{ in } H_{T_0}^{s+1}(x, \theta_p).$$

Proof. We apply Lemma 2.29 to initial data $\{a_{p,l}, b_{p,l} + \partial_t a_{p,l}\}$ to obtain the corresponding solution of (2.104), denoted $\varsigma_{p,l} \in H_D^{s+1}$, and subsequently to each $\{a_{p,l}^n, b_{p,l}^n + \partial_t a_{p,l}^n\}$, obtaining solutions $\varsigma_{p,l}^n \in H_D^{s+1}$. Now we fix a smooth cutoff function $\chi(x_d)$ supported in $[0, D]$, and define

$$(2.111) \quad \sigma_{p,l}(t, x'', \theta_p) = \chi(x_d) \varsigma_{p,l}(t - x_d, x'', \theta_p),$$

$$(2.112) \quad \sigma_{p,l}^n(t, x'', \theta_p) = \chi(x_d) \varsigma_{p,l}^n(t - x_d, x'', \theta_p).$$

It is easy to check that then $\sigma_{p,l}^n, \sigma_{p,l}$ also belong to H_D^{s+1} . Since $\varsigma_{p,l}|_{x_d=0}, \varsigma_{p,l}^n|_{x_d=0}$ are supported in $t > 0$, by finite speed of propagation for solutions to the wave equation (2.104), for any fixed $r > 0$, the (t, y, θ_p) -supports of $\varsigma_{p,l}|_{x_d=r}, \varsigma_{p,l}^n|_{x_d=r}$ are contained in the union of balls of radius r about points in $\{(t, y, \theta_p) : t > 0\}$. Thus, the support is contained in $\{(t, y, \theta_p) : t > -r\}$. It follows that $\sigma_{p,l}(t, y, x_d, \theta_p), \sigma_{p,l}^n(t, y, x_d, \theta_p)$ are zero for all $t \leq 0$, since the supports of $\sigma_{p,l}, \sigma_{p,l}^n$ consist only of points satisfying $t - x_d > -x_d$, i.e. $t > 0$. It is easy to check from (2.111) that the $\sigma_{p,l}, \sigma_{p,l}^n$ have the desired traces at $x_d = 0$, satisfying (2.109). To establish (2.110), first we note we may regard the $\sigma_{p,l}, \sigma_{p,l}^n$ as elements of $H_{T_0}^{s+1}(x, \theta_p)$ by defining them to be equal to zero for $x_d > D$ and restricting to $t < T_0$. One easily verifies the bound

$$(2.113) \quad |\sigma_{p,l}^n - \sigma_{p,l}|_{H_{T_0}^{s+1}} \leq |\sigma_{p,l}^n - \sigma_{p,l}|_{H_D^{s+1}} \leq C_1 |\varsigma_{p,l}^n - \varsigma_{p,l}|_{H_D^{s+1}}.$$

Applying Lemma 2.29 to the solutions $\varsigma_{p,l}^n - \varsigma_{p,l}$, from the above and the estimate (2.107) we get

$$(2.114) \quad |\sigma_{p,l}^n - \sigma_{p,l}|_{H_{T_0}^{s+1}} \leq C_2 (|a_{p,l}^n - a_{p,l}|_{H^{s+1}} + |b_{p,l}^n - b_{p,l}|_{H^s}).$$

Now we conclude (2.110) from Lemma 2.26 and the continuity of the extension map from Lemma 2.28. \square

Now that we have obtained the elliptic profiles, we have determined the ansatz \mathcal{V}^0 .

Definition 2.31. Let $G \in H_{T_0}^{s+1}$ and let $\mathcal{V}_h^0 \in \mathbb{H}_{T_0}^{s+1}$ be the solution of the hyperbolic sub-system constructed in Proposition 2.24, and let the $\mathcal{V}_h^{0,n} \in \mathbb{H}_{T_0}^{s+1}$ be the corresponding iterates. Additionally, let $\sigma_{p,l}, \sigma_{p,l}^n \in H_{T_0}^{s+1}$, where $p \in \mathcal{P} \cup \mathcal{N}$, be the elliptic profiles and iterates obtained in Proposition 2.30.

(i) We define the elliptic part of our ansatz

$$(2.115) \quad \mathcal{V}_e^0(x, \theta_1, \dots, \theta_M) := \sum_{m \in \mathcal{P} \cup \mathcal{N}}^M \sum_{k=1}^{\mu_m} \sigma_{m,k}(x, \theta_m) r_{m,k},$$

and the corresponding n th iterate

$$(2.116) \quad \mathcal{V}_e^{0,n}(x, \theta_1, \dots, \theta_M) := \sum_{m \in \mathcal{P} \cup \mathcal{N}}^M \sum_{k=1}^{\mu_m} \sigma_{m,k}^n(x, \theta_m) r_{m,k}.$$

Our ansatz is defined to be

$$(2.117) \quad \mathcal{V}^0 := \mathcal{V}_h^0 + \mathcal{V}_e^0,$$

and we also define $\mathcal{V}^{0,n} := \mathcal{V}_h^{0,n} + \mathcal{V}_e^{0,n}$.

(ii) We isolate the error in the iterated interior profile equations, defining $R^n(x, \theta)$ by

$$(2.118) \quad R^n := \mathbb{E}(\tilde{L}(\partial_x) \mathcal{V}^{0,n} + \mathcal{M}(\mathcal{V}^{0,n-1}) \partial_\theta \mathcal{V}^{0,n} - F(0) \mathcal{V}^{0,n-1}).$$

Proposition 2.32. Suppose the hypotheses of Definition 2.31 are satisfied. Then $\mathcal{V}^0 \in \mathbb{H}_{T_0}^{s+1}$, and

$$(2.119) \quad \lim_{n \rightarrow \infty} \mathcal{V}^{0,n} = \mathcal{V}^0 \text{ in } \mathbb{H}_{T_0}^s.$$

Proof. The claims follow directly from Proposition 2.24 and Proposition 2.30. \square

Remark 2.33. Recalling (2.81), note that

$$(2.120) \quad \mathbb{E}_h(\tilde{L}(\partial_x) \mathcal{V}^{0,n} + \mathcal{M}(\mathcal{V}^{0,n-1}) \partial_\theta \mathcal{V}^{0,n} - F(0) \mathcal{V}^{0,n-1}) = 0,$$

and so R^n of (2.118) is purely elliptic in the sense that $\mathbb{E}_e R^n = R^n$. Moreover,

$$(2.121) \quad R^n = \sum_{p \in \mathcal{P} \cup \mathcal{N}} \sum_{l=1}^{\mu_p} (X_{\phi_p} \sigma_{p,l}^n - f_{p,l}^n) r_{p,l},$$

which is zero at the boundary $x_d = 0$, since we have satisfied (2.91) for $x_d = 0$. Summarizing the results of Proposition 2.24 and Proposition 2.30, we conclude

$$(2.122) \quad \begin{aligned} & a) \mathbb{E} \mathcal{V}^{0,n} = \mathcal{V}^{0,n} \\ & b) \mathbb{E} \left(\tilde{L}(\partial_x) \mathcal{V}^{0,n} + \mathcal{M}(\mathcal{V}^{0,n-1}) \partial_\theta \mathcal{V}^{0,n} \right) = \mathbb{E}(F(0) \mathcal{V}^{0,n-1}) + R^n \text{ in } x_d \geq 0 \\ & c) B(0) \mathcal{V}^{0,n}(x', 0, \theta_0, \dots, \theta_0) = G(x', \theta_0) \\ & d) \mathcal{V}^{0,n} = 0 \text{ in } t < 0. \end{aligned}$$

The sense in which an error such as R^n is small is clarified by Proposition 2.40.

Remark 2.34. Recall the discussion in Remark 1.10(ii) which gives the bijection between the eigenvalues $\underline{\omega}_m$ with $m \in \mathcal{P}$ and $\underline{\omega}_{m'} = \overline{\underline{\omega}_m}$, $m' \in \mathcal{N}$ and between the corresponding eigenvectors. A careful look at the profile equations shows that the equations for the elliptic profiles come in conjugate pairs. That is, the equation for $\sigma_{m,k}$, some $m \in \mathcal{P}$, is the conjugate of the equation for $\sigma_{m',k'}$ with the corresponding $m' \in \mathcal{N}$ and k' . As a result the solutions we have obtained, the elliptic profiles, also come in conjugate pairs, satisfying $\sigma_{m',k'} = \overline{\sigma_{m,k}}$. Meanwhile, the hyperbolic part of the solution \mathcal{V}_h^0 is real. It follows from these observations and the definition of our ansatz that when we plug in $\theta = \theta(\theta_0, \xi_d)$, getting

$$(2.123) \quad \mathcal{U}^0(x, \theta_0, \xi_d) := \mathcal{V}^0(x, \theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d),$$

we have a function $\mathcal{U}^0(x, \theta_0, \xi_d)$ which is in fact real on $\overline{\mathbb{R}}_+^{d+1} \times \mathbb{T} \times \overline{\mathbb{R}}_+$.

2.5 The expansions for the approximate solution

For this study, we define the spaces E_T^s and \mathcal{E}_T^s as in [2].

$$(2.124) \quad E_T^s = C(x_d, H_T^s(x', \theta_0)) \cap L^2(x_d, H_T^{s+1}(x', \theta_0)),$$

where by $C(x_d, H_T^s(x', \theta_0))$ we actually refer to functions in $C(x_d, H_T^s(x', \theta_0))$ with x_d -support in $[0, D]$ for some large enough D , and for $C(x_d, H_T^s(x', \theta_0))$ we use the $L^\infty(x_d, H_T^s(x', \theta_0))$ norm where the supremum is taken over $x_d \geq 0$. These spaces are algebras and are contained in L^∞ for $s > \frac{d+1}{2}$. Theorem 7.1 of [11], as discussed in Section 1.1.3, regarding existence of solutions of the singular system, tells us that for s large enough, one has existence of solutions to (1.36) in the space E_T^s on a time interval $[0, T]$ independent of $\epsilon \in (0, \epsilon_0]$. For these reasons, the space E_T^s and related estimates are key to our analysis, in particular for our main theorem (Theorem 2.41) which relies on Proposition 2.43, also proved in [11]. Additionally we use the spaces

$$(2.125) \quad \mathcal{E}_T^s = \{\mathcal{U}(x, \theta_0, \xi_d) : \sup_{\xi_d \geq 0} |\mathcal{U}(\cdot, \cdot, \xi_d)|_{E_T^s} < \infty\},$$

which also play a role in Theorem 2.41. The proof of Theorem 2.41 uses estimates regarding functions such as \mathcal{V} in H_T^{s+1} and the corresponding \mathcal{U} in \mathcal{E}_T^s which results from the substitution $\theta = \theta(\theta_0, \xi_d)$. While this moves us out of the space of periodic profiles $H_T^{s+1}(x, \theta)$, we get functions in $\mathcal{E}_T^s(x, \theta_0, \xi_d)$ which we can approximate with finite trigonometric polynomials (truncated expansions¹³) in $\theta = \theta(\theta_0, \xi_d)$.

Definition 2.35. For $k = 1, 2$ we define

$$(2.126) \quad \mathcal{E}_T^{s;k} := \{\mathcal{U}(x, \theta_0, \xi_d) = \mathcal{V}(x, \theta)|_{\theta=\theta(\theta_0, \xi_d)} : \mathcal{V} \in H_T^{s+1;k}\},$$

with the norm $|\cdot|_{\mathcal{E}_T^s}$. (Note: The subscript T has the same indication as it did for the $H_T^{s+1;k}$ used in [2], where the subscript T is at first ignored.)

It is verified in Proposition 2.39 that elements of $\mathcal{E}_T^{s;2}$ are bounded in the \mathcal{E}_T^s norm.

Convergence of expansions in \mathcal{E}_T^s .

The following notation gives us a way to sort the spectra $\alpha \in Z^{M;2}$ of elements \mathcal{V} in $H_T^{s+1;2}$, which will aid in showing $\mathcal{U}(x, \theta_0, \xi_d) = \mathcal{V}(x, \theta)|_{\theta=\theta(\theta_0, \xi_d)}$ has an expansion converging in \mathcal{E}_T^s , with Proposition 2.39.

Definition 2.36. Let $\{z_k\}_{k=1}^\infty$ enumerate the set $\{\alpha \cdot \underline{\omega} : \alpha \in Z^{M;2}\}$. For each j, k , we define

$$(2.127) \quad C_{j,k} := \{\alpha \in Z^{M;2} : \sum_{i=1}^M \alpha_i = j, \alpha \cdot \underline{\omega} = z_k\}$$

and pick out one element $\alpha_{(j,k)} \in C_{j,k}$.

¹³These expansions are given by Proposition 2.39.

Remark 2.37. A nice consequence of the fact that we work with $Z^{M;2}$ instead of general $Z^{M;k}$ is that we can easily show the $C_{j,k}$ are finite¹⁴. To see this, first fix $j \in \mathbb{Z}$, $k \in \{1, 2, \dots\}$, and $p, q \in \{1, \dots, M\}$ and take arbitrary $\alpha \in C_{j,k}$ such that all but the p th and q th components are zero (either of the p th and q th components may be zero, as well.) We claim this is the only such element of $C_{j,k}$. This is because $\underline{\omega}_p \neq \underline{\omega}_q$ implies

$$(2.128) \quad \begin{pmatrix} 1 & 1 \\ \underline{\omega}_p & \underline{\omega}_q \end{pmatrix} \begin{pmatrix} \alpha_p \\ \alpha_q \end{pmatrix} = \begin{pmatrix} j \\ z_k \end{pmatrix}$$

has a unique solution (α_p, α_q) . Thus, the number of such α in $C_{j,k}$ is bounded by the number of pairs of components, so $|C_{j,k}| \leq M(M-1)/2$.

A function $\mathcal{V}(x, \theta) \in H_T^{s+1;2}(x, \theta)$ of $(x, \theta) \in \mathbb{R}^{d+1} \times \mathbf{C}^M$ has a series

$$(2.129) \quad \mathcal{V}(x, \theta) = \sum_{\alpha \in Z^{M;2}} V_\alpha(x) e^{i\alpha \cdot \theta}$$

and, for fixed x_d , squared $H_T^s(x', \theta)$ norm

$$(2.130) \quad |\mathcal{V}(x, \theta)|_{H_T^s(x', \theta)}^2 = \sum_{\alpha \in Z^{M;2}} \sum_{|\beta| \leq s} |\partial_{x'}^\beta V_\alpha(x)|_{L^2(x')}^2 (1 + |\alpha|)^{2(s-|\beta|)}.$$

From Sobolev embedding and the fact that

$$(2.131) \quad H_T^{s+1}(x, \theta) \subset L^2(x_d, H_T^{s+1}(x', \theta)) \cap H^1(x_d, H_T^s(x', \theta)),$$

we find $\mathcal{V}(x, \theta) \in L^2(x_d, H_T^{s+1}(x', \theta)) \cap C(x_d, H_T^s(x', \theta))$, implying the partial sums of the series (2.129) are bounded and converge in $H_T^s(x', \theta)$ uniformly with respect to $x_d \geq 0$. We will prove Proposition 2.39 by using these facts with the following lemma, which shows for finite truncations of expansions (2.129) that the norm $|\cdot|_{H_T^s(x', \theta)}$ dominates $|\cdot|_{\theta=\theta(\theta_0, \xi_d)}|_{H_T^s(x', \theta_0)}$ independent of (x_d, ξ_d) .

Lemma 2.38. Suppose $\mathcal{V} \in H_T^{s+1;2}(x, \theta)$ with series given by (2.129). For $\theta(\theta_0, \xi_d)$ as in (2.10) and integers $M_1 \leq M_2$, $0 < N_1 \leq N_2$, we have the following inequality:

$$(2.132) \quad \left| \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \sum_{\alpha \in C_{j,k}} V_\alpha(x) e^{i\alpha \cdot \theta(\theta_0, \xi_d)} \right|_{H_T^s(x', \theta_0)}^2 \leq \left| \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \sum_{\alpha \in C_{j,k}} V_\alpha(x) e^{i\alpha \cdot \theta} \right|_{H_T^s(x', \theta)}^2.$$

Proof. We estimate

$$(2.133) \quad \left| \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \sum_{\alpha \in C_{j,k}} V_\alpha(x) e^{i\alpha \cdot \theta(\theta_0, \xi_d)} \right|_{H_T^s(x', \theta_0)}^2 = \left| \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \left(\sum_{\alpha \in C_{j,k}} V_\alpha(x) e^{i\alpha \cdot \omega \xi_d} \right) e^{ij\theta_0} \right|_{H_T^s(x', \theta_0)}^2,$$

$$(2.134) \quad = \sum_{j=M_1}^{M_2} \sum_{|\beta| \leq s} \left| \left(\sum_{k=N_1}^{N_2} \sum_{\alpha \in C_{j,k}} \partial_{x'}^\beta V_\alpha(x) e^{i\alpha \cdot \omega \xi_d} \right) \right|_{L^2(x')}^2 (1 + |j|)^{2(s-|\beta|)},$$

$$(2.134) \quad \leq \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \sum_{\alpha \in C_{j,k}} \sum_{|\beta| \leq s} |\partial_{x'}^\beta V_\alpha(x) e^{i\alpha \cdot \omega \xi_d}|_{L^2(x')}^2 (1 + |j|)^{2(s-|\beta|)}.$$

For (2.133), we used the formula which that in (2.130) generalizes. For $\alpha \in C_{j,k}$, we have $|j| = |\sum_i \alpha_i| \leq |\alpha|$ and $\text{Im}(\alpha \cdot \omega) \geq 0$, so for all $\xi_d \geq 0$, the sum in (2.134) is bounded by

$$(2.135) \quad \left| \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \sum_{\alpha \in C_{j,k}} V_\alpha(x) e^{i\alpha \cdot \theta} \right|_{H_T^s(x', \theta)}^2 = \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \sum_{\alpha \in C_{j,k}} \sum_{|\beta| \leq s} |\partial_{x'}^\beta V_\alpha(x)|_{L^2(x')}^2 (1 + |\alpha|)^{2(s-|\beta|)},$$

where the equality (2.135) follows from the formula in (2.130). \square

¹⁴We work with $Z^{M;2}$ due to the fact that we only have to consider quadratic interactions. Further discussion on interactions and resonances can be found in Sections 2.1 and 2.2.

With the estimate of Lemma 2.38, we are ready to prove that elements of $H_T^{s+1;2}$ yield elements with expansions converging in \mathcal{E}_T^s upon the substitution $\theta = \theta(\theta_0, \xi_d)$.

Proposition 2.39. *Let $\mathcal{V} \in H_T^{s+1;2}(x, \theta)$ with expansion (2.129) and set $\mathcal{U} = \mathcal{V}|_{\theta=\theta(\theta_0, \xi_d)}$ for $\theta(\theta_0, \xi_d)$ as in (2.10). Then we have $\mathcal{U} \in \mathcal{E}_T^s$:*

$$(2.136) \quad |\mathcal{U}|_{\mathcal{E}_T^s} \leq C|\mathcal{V}|_{H_T^{s+1}},$$

and we have convergence in \mathcal{E}_T^s to \mathcal{U} of finite partial sums, independent of arrangement, of the series

$$(2.137) \quad \mathcal{U}(x, \theta_0, \xi_d) = \sum_{\alpha \in Z^{M;2}} V_\alpha(x) e^{i\alpha \cdot \theta(\theta_0, \xi_d)}.$$

Proof of Proposition 2.39. We explicitly show the convergence of the finite partial sums

$$(2.138) \quad \sum_{\substack{-n \leq j \leq n \\ 1 \leq k \leq n}} \sum_{\alpha \in C_{j,k}} V_\alpha(x) e^{i\alpha \cdot \theta(\theta_0, \xi_d)}.$$

For integers n_1, n_2 with $n_1 \leq n_2$, an application of Lemma 2.38 gives

$$(2.139) \quad \left| \sum_{\substack{n_1 \leq |j| \leq n_2 \\ n_1 \leq k \leq n_2}} \sum_{\alpha \in C_{j,k}} V_\alpha(x) e^{i\alpha \cdot \theta(\theta_0, \xi_d)} \right|_{H_T^s(x', \theta_0)}^2 \leq \left| \sum_{\substack{n_1 \leq |j| \leq n_2 \\ n_1 \leq k \leq n_2}} \sum_{\alpha \in C_{j,k}} V_\alpha(x) e^{i\alpha \cdot \theta} \right|_{H_T^s(x', \theta)}^2.$$

It follows from (2.139) that since the sequence of the partial sums

$$(2.140) \quad \sum_{\substack{-n \leq j \leq n \\ 1 \leq k \leq n}} \sum_{\alpha \in C_{j,k}} V_\alpha(x) e^{i\alpha \cdot \theta}$$

is Cauchy in $H_T^s(x', \theta)$ uniformly with respect to (x_d, ξ_d) , so is the sequence (2.138) in $H_T^s(x', \theta_0)$, and thus we have convergence. We similarly get convergence of the (2.138) in $L^2(x_d, H_T^{s+1}(x', \theta_0))$ after integrating the inequality (2.139) (with $s+1$ in place of s) with respect to x_d and noting the right hand side is independent of ξ_d . Hence, the (2.138) converge in \mathcal{E}_T^s . It is not hard to show the limit is in fact $\mathcal{V}|_{\theta=\theta(\theta_0, \xi_d)}$, and the estimate (2.136) easily follows. The same proof works for arbitrarily ordered sums, where one instead considers finite $\mathcal{B}_n \nearrow Z^{M;2}$ and similarly gets that the sequence of the

$$(2.141) \quad \sum_{\alpha \in \mathcal{B}_n} V_\alpha(x) e^{i\alpha \cdot \theta(\theta_0, \xi_d)}$$

is Cauchy in the desired space because the sequence of the $\sum_{\alpha \in \mathcal{B}_n} V_\alpha(x) e^{i\alpha \cdot \theta}$ is Cauchy. \square

2.6 Error analysis

The following proposition is used to make precise the notion that $\sigma_{p,l}(x, \theta_p)$ and $\sigma_{p,l}^n(x, \theta_p)$ are approximate solutions of (2.84) and (2.85)-(2.88), respectively.

Proposition 2.40. *Let $R(x, \theta) \in H_T^{s;1}(x, \theta)$ have the property that it is polarized by the elliptic projector \mathbb{E}_e defined in (2.23), i.e. that $\mathbb{E}_e(R) = R$, and suppose*

$$(2.142) \quad R|_{x_d=0} = 0.$$

Then

$$(2.143) \quad \lim_{\epsilon \rightarrow 0} |R(x, \theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d)|_{\xi_d = \frac{x_d}{\epsilon}}|_{E_T^{s-1}(x, \theta_0)} = 0.$$

Proof. It suffices to consider the case $R(x, \theta_p) \in H_T^s(x, \theta_p)$, $\mathbb{E}_p(R) = R$, for some $p \in \mathcal{P} \cup \mathcal{N}$, and show

$$(2.144) \quad \lim_{\epsilon \rightarrow 0} |R(x, \theta_0 + \underline{\omega}_p \xi_d)|_{\xi_d = \frac{x_d}{\epsilon}}|_{E_T^{s-1}(x, \theta_0)} = 0.$$

First, we show

$$(2.145) \quad \lim_{\epsilon \rightarrow 0} \sup_{x_d \geq 0} |R(x, \theta_0 + \underline{\omega}_p \xi_d)|_{\xi_d = \frac{x_d}{\epsilon}}|_{H_T^{s-1}(x', \theta_0)} = 0.$$

Note R has an expansion of the form

$$(2.146) \quad R(x, \theta_p) = \sum_{j \in Z_p \setminus 0} a_j(x) e^{ij\theta_p}.$$

Thus, noting $R(x, \theta_0) \in H_T^s(x, \theta_0) \subset C(x_d, H_T^{s-1}(x', \theta_0))$, fixing x_d , we get the norm

$$(2.147) \quad |R(x, \theta_p)|_{H_T^{s-1}(x', \theta_p)}^2 = \sum_{j \in Z_p \setminus 0} \sum_{|\beta| \leq s-1} |\partial_{x'}^\beta a_j(x)|_{L^2(x')}^2 (1 + |j|)^{2(s-1-|\beta|)}.$$

We let $\xi_d = \frac{x_d}{\epsilon}$ and find

$$(2.148) \quad R(x, \theta_0 + \underline{\omega}_p \xi_d) = \sum_{j \in Z_p \setminus 0} (e^{-\text{Im}(j\underline{\omega}_p)\xi_d} e^{i\text{Re}(j\underline{\omega}_p)\xi_d} a_j(x)) e^{ij\theta_0}.$$

Then it follows

$$(2.149) \quad \begin{aligned} |R(x, \theta_0 + \underline{\omega}_p \xi_d)|_{H_T^{s-1}(x', \theta_0)}^2 &= \sum_{j \in Z_p \setminus 0} \sum_{|\beta| \leq s-1} (e^{-\text{Im}(j\underline{\omega}_p)\xi_d})^2 |\partial_{x'}^\beta a_j(x)|_{L^2(x')}^2 (1 + |j|)^{2(s-1-|\beta|)} \\ &\leq e^{-2|\text{Im}\underline{\omega}_p|\xi_d} \sum_{j \in Z_p \setminus 0} \sum_{|\beta| \leq s-1} |\partial_{x'}^\beta a_j(x)|_{L^2(x')}^2 (1 + |j|)^{2(s-1-|\beta|)} \end{aligned}$$

$$(2.150) \quad = e^{-2|\text{Im}\underline{\omega}_p|\xi_d} |R(x, \theta_0)|_{H_T^{s-1}(x', \theta_0)}^2$$

Now we show that, as ϵ tends to zero,

$$(2.151) \quad \sup_{x_d \in [0, \sqrt{\epsilon}]} |R(x, \theta_0 + \underline{\omega}_p \frac{x_d}{\epsilon})|_{H_T^{s-1}(x', \theta_0)} \rightarrow 0,$$

and

$$(2.152) \quad \sup_{x_d \geq \sqrt{\epsilon}} |R(x, \theta_0 + \underline{\omega}_p \frac{x_d}{\epsilon})|_{H_T^{s-1}(x', \theta_0)} \rightarrow 0.$$

Set $h(x_d) = R(x', x_d, \theta_0) \in H_T^{s-1}(x', \theta_0)$. To see (2.151), observe that the term is bounded by

$$(2.153) \quad \sup_{x_d \in [0, \sqrt{\epsilon}]} e^{-2|\text{Im}\underline{\omega}_p|\frac{x_d}{\epsilon}} |h(x_d)|_{H_T^{s-1}(x', \theta_0)}^2 \leq \sup_{x_d \in [0, \sqrt{\epsilon}]} |h(x_d)|_{H_T^{s-1}(x', \theta_0)}^2,$$

which converges to 0 as ϵ tends to zero, since $h \in C(x_d, H_T^{s-1}(x', \theta_0))$ with $h(0) = 0$. That (2.152) holds follows from the fact that this term is bounded by

$$(2.154) \quad \sup_{x_d \geq \sqrt{\epsilon}} e^{-2|\text{Im}\underline{\omega}_p|\frac{x_d}{\epsilon}} |h(x_d)|_{H_T^{s-1}(x', \theta_0)}^2 \leq \sup_{x_d \geq 0} |h(x_d)|_{H_T^{s-1}(x', \theta_0)}^2 e^{-2|\text{Im}\underline{\omega}_p|\frac{1}{\sqrt{\epsilon}}},$$

which also converges to 0 as ϵ tends to zero. Now we check that

$$(2.155) \quad \lim_{\epsilon \rightarrow 0} |R(x, \theta_0 + \underline{\omega}_p \xi_d)|_{\xi_d = \frac{x_d}{\epsilon}}|_{L^2(x_d, H_T^s(x', \theta_0))} = 0.$$

It is not hard to show that

$$(2.156) \quad |R(x, \theta_0 + \underline{\omega}_p \frac{x_d}{\epsilon})|_{L^2(x_d, H_T^s(x', \theta_0))}^2 \leq |e^{-|\text{Im}\underline{\omega}_p|\frac{x_d}{\epsilon}}|_{L^2} |R(x, \theta_0)|_{L^2(x_d, H_T^s(x', \theta_0))}^2 \leq D\sqrt{\epsilon},$$

which completes the proof. \square

Now we are ready to show that, as $\epsilon \rightarrow 0$, the approximate solution u_ϵ^a converges to the exact solution u_ϵ in L^∞ . This is a corollary of the following theorem.

Theorem 2.41. *For $M_0 = 2(d+2) + 1$ and $s \geq 1 + [M_0 + \frac{d+1}{2}]$ let $G(x', \theta_0) \in H_T^{s+1}$ have compact support in x' and vanish in $t \leq 0$. Let $U_\epsilon(x, \theta_0) \in E_{T_0}^s$ be the exact solution to the singular system for $0 < \epsilon \leq \epsilon_0$ given by Theorem 7.1 of [11], let $\mathcal{V}^0 \in \mathbb{H}_{T_0}^{s+1}$ be the profile given by (1.30), and let $\mathcal{U}^0 \in \mathcal{E}_{T_0}^s$ be defined by*

$$(2.157) \quad \mathcal{U}^0(x, \theta_0, \xi_d) = \mathcal{V}^0(x, \theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d).$$

Here $0 < T_0 \leq T$ is the minimum of the existence times for the quasilinear problems (1.36) and (2.41). Define

$$(2.158) \quad \mathcal{U}_\epsilon^0(x, \theta_0) := \mathcal{U}^0(x, \theta_0, \frac{x_d}{\epsilon}).$$

The family \mathcal{U}_ϵ^0 is uniformly bounded in $E_{T_0}^s$ for $0 < \epsilon \leq \epsilon_0$ and satisfies

$$(2.159) \quad |U_\epsilon - \mathcal{U}_\epsilon^0|_{E_{T_0}^{s-1}} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Upon evaluating our placeholder $\xi_d = \frac{x_d}{\epsilon}$ in the argument of $\mathcal{U}^0 \in \mathcal{E}_T^s$, we get the function \mathcal{U}_ϵ^0 in E_T^s , the same space containing the solutions of the singular system. Indeed, the E_T^s norm is that of the estimate of Proposition 2.43, key to our proof of Theorem 2.41, and the norm in which we show our approximate solution \mathcal{U}_ϵ^0 is close to the exact solution of the singular system.

Lemma 2.42. *For $m \geq 0$ suppose $\mathcal{V}(x, \theta) \in \mathbb{H}_T^{m+1}$, $\mathbb{E}\mathcal{V} = \mathcal{V}$, and set $\mathcal{U}(x, \theta_0, \xi_d) = \mathcal{V}(x, \theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d)$. Then, setting $\mathcal{U}_\epsilon(x, \theta_0) = \mathcal{U}(x, \theta_0, \frac{x_d}{\epsilon})$,*

$$(2.160) \quad |\mathcal{U}_\epsilon|_{E_T^m} \leq |\mathcal{U}|_{\mathcal{E}_T^m}.$$

Proof. The proof is almost identical to the argument used in [2] to prove Lemma 2.25 (b) with Lemma 2.7. \square

Proof of Theorem 2.41. It suffices to prove boundedness of the family \mathcal{U}_ϵ^0 in $E_{T_0}^s$ along with the following three statements:

$$(2.161) \quad \begin{aligned} (a) \quad & \lim_{n \rightarrow \infty} U_\epsilon^n = U_\epsilon \text{ in } E_{T_0}^{s-1} \text{ uniformly with respect to } \epsilon \in (0, \epsilon_0] \\ (b) \quad & \lim_{n \rightarrow \infty} \mathcal{U}_\epsilon^{0,n} = \mathcal{U}_\epsilon^0 \text{ in } E_{T_0}^{s-1} \text{ uniformly with respect to } \epsilon \in (0, \epsilon_0] \\ (c) \quad & \text{For each } n \quad \lim_{\epsilon \rightarrow 0} |U_\epsilon^n - \mathcal{U}_\epsilon^{0,n}|_{E_{T_0}^{s-1}} = 0. \end{aligned}$$

The uniform boundedness of U_ϵ and the U_ϵ^n in $E_{T_0}^s$ and that (2.161)(a) holds are proved in [11], Theorem 7.1, with the use of the iteration scheme (1.37) and the following linear estimate:

Proposition 2.43 ([11], Cor. 7.2). *Let $s \geq [M_0 + \frac{d+1}{2}]$ and consider the problem (1.37), where $G \in H_T^{s+1}$ has compact support and vanishes in $t \leq 0$, and where the right side of (1.37)(a) is replaced by $\mathcal{F} \in \mathbb{E}_T^s$ with $\text{supp } \mathcal{F} \subset \{t \geq 0, 0 \leq x_d \leq E\}$. Suppose $U_\epsilon^n \in E_T^s$ has compact x -support and that for some $K > 0$, $\epsilon_1 > 0$ we have*

$$(2.162) \quad |U_\epsilon^n|_{E_T^s} + |\epsilon \partial_{x_d} U_\epsilon^n|_{L^\infty} \leq K \text{ for } \epsilon \in (0, \epsilon_1].$$

Then there exist constants $T_0(K)$ and $\epsilon_0(K) \leq \epsilon_1$ such that for $0 < \epsilon \leq \epsilon_0$ and $T \leq T_0$ we have

$$(2.163) \quad |U_\epsilon^{n+1}|_{E_T^s} + \sqrt{T} \langle U_\epsilon^{n+1} \rangle_{s+1, T} \leq C(K, E) \sqrt{T} (|\mathcal{F}|_{E_T^s} + \langle G \rangle_{s+1, T}).$$

The desired boundedness of \mathcal{U}_ϵ^0 and the $\mathcal{U}_\epsilon^{0,n}$ in $E_{T_0}^s$ follows from the boundedness of \mathcal{V}^0 and the uniform boundedness of the $\mathcal{V}^{0,n}$ in $\mathbb{H}_{T_0}^{s+1}$, considered with Proposition 2.39 and Lemma 2.42. Similarly, from the fact that

$$(2.164) \quad \lim_{n \rightarrow \infty} \mathcal{V}^{0,n} = \mathcal{V}^0 \text{ in } \mathbb{H}_{T_0}^s,$$

we get (2.161)(b).

Now we proceed by proving (2.161)(c). Fix $\delta > 0$. Noting $\mathbb{E}\mathcal{V}^{0,n} = \mathcal{V}^{0,n}$ and $\mathbb{E}\mathcal{V}^{0,n+1} = \mathcal{V}^{0,n+1}$, choose finite partial trigonometric polynomial sums $\mathcal{V}_p^{0,n}$ and $\mathcal{V}_p^{0,n+1}$ of (resp.) $\mathcal{V}^{0,n}$ and $\mathcal{V}^{0,n+1}$ such that

$$(2.165) \quad \begin{aligned} (a) \quad & \mathbb{E}\mathcal{V}_p^{0,n} = \mathcal{V}_p^{0,n} \text{ and } \mathbb{E}\mathcal{V}_p^{0,n+1} = \mathcal{V}_p^{0,n+1}, \\ (b) \quad & |\mathcal{V}^{0,n} - \mathcal{V}_p^{0,n}|_{H_{T_0}^{s+1}} < \delta, |\mathcal{V}^{0,n+1} - \mathcal{V}_p^{0,n+1}|_{H_{T_0}^{s+1}} < \delta, \end{aligned}$$

and note that, as a result of (2.165)(b), we also have

$$(2.166) \quad |\partial_{x_d}\mathcal{V}^{0,n+1} - \partial_{x_d}\mathcal{V}_p^{0,n+1}|_{H_{T_0}^s} < \delta.$$

Observe then, as a consequence of Proposition 2.39, we get

$$(2.167) \quad |\mathcal{U}^{0,n} - \mathcal{U}_p^{0,n}|_{\mathcal{E}_{T_0}^s} < C\delta, |\mathcal{U}^{0,n+1} - \mathcal{U}_p^{0,n+1}|_{\mathcal{E}_{T_0}^s} < C\delta, \text{ and } |\partial_{x_d}\mathcal{U}^{0,n+1} - \partial_{x_d}\mathcal{U}_p^{0,n+1}|_{\mathcal{E}_{T_0}^{s-1}} < C\delta,$$

where $\mathcal{U}_p^{0,n}$ and $\mathcal{U}_p^{0,n+1}$ are obtained from (resp.) $\mathcal{V}_p^{0,n}$ and $\mathcal{V}_p^{0,n+1}$ via the substitution $\theta = \theta(\theta_0, \xi_d)$. The induction assumption is

$$(2.168) \quad \lim_{\epsilon \rightarrow 0} |U_\epsilon^n - \mathcal{U}_\epsilon^{0,n}|_{E_{T_0}^{s-1}} = 0.$$

With the boundedness of the U_ϵ^n in $E_{T_0}^s$, it follows

$$(2.169) \quad \lim_{\epsilon \rightarrow 0} |F(\epsilon U_\epsilon^n)(U_\epsilon^n) - F(0)\mathcal{U}_\epsilon^{0,n}|_{E_{T_0}^{s-1}} = 0.$$

Observe that since $\mathcal{V}^{0,n}$ and $\mathcal{V}_p^{0,n}$ are invariant under \mathbb{E} , Lemma 2.42 applies. Thus, using the estimate (2.160), from (2.169) and (2.167) we obtain

$$(2.170) \quad |F(\epsilon U_\epsilon^n)(U_\epsilon^n) - F(0)\mathcal{U}_{p,\epsilon}^{0,n}|_{E_{T_0}^{s-1}} \leq C\delta + c(\epsilon),$$

where $c(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Now we define

$$(2.171) \quad \mathcal{G}_p = \tilde{L}(\partial_x)\mathcal{V}_p^{0,n+1} + \mathcal{M}(\mathcal{V}_p^{0,n})\partial_\theta\mathcal{V}_p^{0,n+1}.$$

Then

$$(2.172) \quad \begin{aligned} \mathbb{E}(\mathcal{G}_p - F(0)\mathcal{V}_p^{0,n}) &= \mathbb{E}(\tilde{L}(\partial_x)(\mathcal{V}_p^{0,n+1} - \mathcal{V}^{0,n+1})) + \\ &\quad \mathbb{E}(\mathcal{M}(\mathcal{V}_p^{0,n})\partial_\theta\mathcal{V}_p^{0,n+1} - \mathcal{M}(\mathcal{V}^{0,n})\partial_\theta\mathcal{V}^{0,n+1}) + \\ &\quad \mathbb{E}(F(0)(\mathcal{V}^{0,n} - \mathcal{V}_p^{0,n})) + R^{n+1}. \end{aligned}$$

Thus using (2.165)(b) and continuity of both $\mathbb{E} : H^{s;2} \rightarrow H^{s;1}$ and multiplication from $H^{s;1} \times H^{s;1} \rightarrow H^{s;2}$, we get

$$(2.173) \quad |\mathbb{E}\mathcal{G}_p - \mathbb{E}(F(0)\mathcal{V}_p^{0,n}) - R^{n+1}|_{H_{T_0}^s} = O(\delta).$$

We define the operator

$$(2.174) \quad \mathbb{L}_0 = \tilde{L}(\partial_x) + \frac{1}{\epsilon}\tilde{L}(d\phi_0)\partial_{\theta_0} + \mathcal{M}'(\mathcal{U}_{p,\epsilon}^{0,n})\partial_{\theta_0}.$$

We claim

$$(2.175) \quad |\mathbb{L}_0 U_\epsilon^{n+1} - F(\epsilon U_\epsilon^n)U_\epsilon^n|_{E_{T_0}^{s-1}} \leq C\delta + c(\epsilon),$$

where $c(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. This follows from (1.37)(a) and

$$\begin{aligned}
& |\tilde{A}_j(\epsilon U_\epsilon^n) \partial_{x_j} U_\epsilon^{n+1} - \tilde{A}_j(0) \partial_{x_j} U_\epsilon^{n+1}|_{E_{T_0}^{s-1}} = O(\epsilon) \\
(2.176) \quad & \left| \frac{1}{\epsilon} \tilde{A}_j(\epsilon U_\epsilon^n) \beta_j \partial_{\theta_0} U_\epsilon^{n+1} - \left(\frac{1}{\epsilon} \tilde{A}_j(0) \beta_j \partial_{\theta_0} U_\epsilon^{n+1} + \partial_u \tilde{A}_j(0) U_\epsilon^n \beta_j \partial_{\theta_0} U_\epsilon^{n+1} \right) \right|_{E_{T_0}^{s-1}} = O(\epsilon) \\
& \left| \partial_u \tilde{A}_j(0) (U_\epsilon^n - \mathcal{U}_{p,\epsilon}^{0,n}) \beta_j \partial_{\theta_0} U_\epsilon^{n+1} \right|_{E_{T_0}^{s-1}} \leq C |U_\epsilon^n - \mathcal{U}_{p,\epsilon}^{0,n}|_{E_{T_0}^{s-1}} \leq c(\epsilon) + O(\delta).
\end{aligned}$$

Setting $\mathcal{G}'_p = \tilde{L}(\partial_x) \mathcal{U}_p^{0,n+1} + \mathcal{M}'(\mathcal{U}_p^{0,n}) \partial_{\theta_0} \mathcal{U}_p^{0,n+1}$, since $\mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U}_p^{0,n+1} = 0$, we get

$$(2.177) \quad \mathbb{L}_0 \mathcal{U}_{p,\epsilon}^{0,n+1} = \mathcal{G}'_{p,\epsilon}.$$

From now on we use $|\cdot|_{(\theta_0, \xi_d)}$ to denote evaluation at $\theta = \theta(\theta_0, \xi_d)$, and $|\cdot|_{(\theta_0, \xi_d), \epsilon}$ to indicate $|\cdot|_{(\theta_0, \xi_d)}$ followed by evaluation at $\xi_d = \frac{x_d}{\epsilon}$. It is easy to check $\mathcal{G}'_p = \mathcal{G}_p|_{(\theta_0, \xi_d)}$, particularly because $\mathcal{V}_p^{0,n}, \mathcal{V}_p^{0,n+1}$ are finite polynomials. Thus

$$\begin{aligned}
(2.178) \quad \mathbb{L}_0 \mathcal{U}_{p,\epsilon}^{0,n+1} - F(0) \mathcal{U}_{p,\epsilon}^{0,n} &= \mathcal{G}'_{p,\epsilon} - F(0) \mathcal{U}_{p,\epsilon}^{0,n} \\
&= [\mathcal{G}_p - F(0) \mathcal{V}_p^{0,n}]|_{(\theta_0, \xi_d), \epsilon}
\end{aligned}$$

$$(2.179) \quad = [(\mathbb{E}(\mathcal{G}_p - F(0) \mathcal{V}_p^{0,n}) - R^{n+1}) + R^{n+1} + (I - \mathbb{E})(\mathcal{G}_p - F(0) \mathcal{V}_p^{0,n})]|_{(\theta_0, \xi_d), \epsilon}$$

We claim:

(i)

$$(2.180) \quad |(\mathbb{E}(\mathcal{G}_p - F(0) \mathcal{V}_p^{0,n}) - R^{n+1})|_{(\theta_0, \xi_d), \epsilon}|_{E_{T_0}^{s-1}} = O(\delta),$$

(ii)

$$(2.181) \quad \lim_{\epsilon \rightarrow 0} |R^{n+1}|_{(\theta_0, \xi_d), \epsilon}|_{E_{T_0}^{s-1}(x, \theta_0)} = 0,$$

and (iii) there exists $\mathcal{V}_p^1 \in H_{T_0}^{s;2}$ such that for $\mathcal{U}_p^1 = \mathcal{V}_p^1|_{\theta(\theta_0, \xi_d)}$,

$$(2.182) \quad \mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U}_p^1 = -[(I - \mathbb{E})(\mathcal{G}_p - F(0) \mathcal{V}_p^{0,n})]|_{\theta(\theta_0, \xi_d)}.$$

(i) follows from (2.173) followed by application of Proposition 2.39 and then Lemma 2.42.

To see (ii), observe it is clear from our definition of R^{n+1} that the continuity of \mathbb{E} implies $R^{n+1}(x, \theta) \in H_{T_0}^{s;1}(x, \theta)$, and recall from Remark 2.33 that we have $\mathbb{E}_e R^{n+1} = R^{n+1}$ and (2.121). By requiring (2.95), and thus (2.92), to hold at $x_d = 0$, we have obtained $R^{n+1}|_{x_d=0} = 0$. So Proposition 2.40 grants us (2.181).

Regarding (iii), note that, by Remark 2.8, (2.182) holds if and only if

$$(2.183) \quad \mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U}_p^1 = -[(I - \mathbb{E}^b)(\mathcal{G}_p - F(0) \mathcal{V}_p^{0,n})]|_{\theta(\theta_0, \xi_d)},$$

for which a sufficient condition is that, where $\mathcal{U}_p^1 = \mathcal{V}_p^1|_{\theta(\theta_0, \xi_d)}$,

$$(2.184) \quad \mathcal{L}(\partial_\theta) \mathcal{V}_p^1 = -(I - \mathbb{E}^b)(\mathcal{G}_p - F(0) \mathcal{V}_p^{0,n}).$$

Proposition 2.9 guarantees the existence of such \mathcal{V}_p^1 , so proof of (iii) is complete.

Noting

$$(2.185) \quad \mathbb{L}_0(\epsilon \mathcal{U}_{p,\epsilon}^1) = (\mathcal{L}'(\partial_{\theta_0}, \partial_{\xi_d}) \mathcal{U}_p^1)_\epsilon + (\tilde{L}(\partial_x) \epsilon \mathcal{U}_p^1)_\epsilon + \mathcal{M}'(\mathcal{U}_{p,\epsilon}^{0,n}) \partial_{\theta_0}(\epsilon \mathcal{U}_{p,\epsilon}^1),$$

it follows from (2.179)-(2.182) that

$$(2.186) \quad |\mathbb{L}_0(\mathcal{U}_{p,\epsilon}^{0,n+1} + \epsilon \mathcal{U}_{p,\epsilon}^1) - F(0) \mathcal{U}_{p,\epsilon}^{0,n}|_{E_{T_0}^{s-1}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon,$$

where $c(\epsilon)$ has been altered, but still tends to zero as $\epsilon \rightarrow 0$. It follows from equations (2.170), (2.175), and (2.186) that

$$(2.187) \quad |\mathbb{L}_0(U_\epsilon^{n+1} - (\mathcal{U}_{p,\epsilon}^{0,n+1} + \epsilon \mathcal{U}_{p,\epsilon}^1))|_{E_{T_0}^{s-1}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon.$$

Now we claim that we have the following estimates:

$$(2.188) \quad \begin{aligned} (a) \quad & \left| \left(\partial_{x_d} + \mathbb{A}(\epsilon \mathcal{U}_{p,\epsilon}^{0,n}, \partial_{x'} + \frac{\beta \cdot \partial_{\theta_0}}{\epsilon}) \right) (U_\epsilon^{n+1} - (\mathcal{U}_{p,\epsilon}^{0,n+1} + \epsilon \mathcal{U}_{p,\epsilon}^1)) \right|_{E_{T_0}^{s-1}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon \\ (b) \quad & |B(\epsilon \mathcal{U}_{p,\epsilon}^{0,n}) (U_\epsilon^{n+1} - (\mathcal{U}_{p,\epsilon}^{0,n+1} + \epsilon \mathcal{U}_{p,\epsilon}^1))|_{H_{T_0}^s} \leq C\delta + c(\epsilon) + K(\delta)\epsilon. \end{aligned}$$

That (2.188)(a) holds follows from (2.187) with the use of estimates similar to (2.176), and (2.188)(b) follows from the boundary conditions (1.36)(b) and (2.122)(c) together with Proposition 2.39 and Lemma 2.42. After applying the estimate of Proposition 2.43, we get that

$$(2.189) \quad |U_\epsilon^{n+1} - (\mathcal{U}_{p,\epsilon}^{0,n+1} + \epsilon \mathcal{U}_{p,\epsilon}^1)|_{E_{T_0}^{s-1}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon.$$

from which we conclude

$$(2.190) \quad |U_\epsilon^{n+1} - \mathcal{U}_\epsilon^{0,n+1}|_{E_{T_0}^{s-1}} \leq C\delta + c(\epsilon) + K(\delta)\epsilon.$$

This finishes the induction step, completing the proof of the theorem. □

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